

# SINGULAR COHOMOLOGY OF THE ANALYTIC MILNOR FIBER, AND MIXED HODGE STRUCTURE ON THE NEARBY COHOMOLOGY

JOHANNES NICAISE

ABSTRACT. We describe the homotopy type of the analytic Milnor fiber in terms of a strictly semi-stable model, and we show that its singular cohomology coincides with the weight zero part of the mixed Hodge structure on the nearby cohomology. We give a similar expression for Denef and Loeser's motivic Milnor fiber in terms of a strictly semi-stable model.

MSC 2000: 32S30, 32S55, 14D07, 14G22

## 1. INTRODUCTION

Let  $k$  be any field, and let  $X$  be a  $k$ -variety, endowed with a morphism  $f : X \rightarrow \operatorname{Spec} k[t]$ . Let  $x$  be a closed point on the special fiber  $X_s = f^{-1}(0)$ . The analytic Milnor fiber  $\mathcal{F}_x$  of  $f$  at  $x$  was introduced and studied by Julien Sebag and the author in [20, 21, 22]. The object  $\mathcal{F}_x$  is defined as the generic fiber of the special formal scheme

$$\widehat{f} : \operatorname{Spf} \widehat{\mathcal{O}}_{X,x} \rightarrow \operatorname{Spf} k[[t]]$$

It is an analytic space over the field of Laurent series  $k((t))$ , and serves as a non-archimedean model of the classical topological Milnor fibration (for  $k = \mathbb{C}$  and  $X$  smooth). The arithmetic and geometric properties of  $\mathcal{F}_x$  reflect the nature of the singularity of  $f$  at  $x$ . For instance, the  $\ell$ -adic cohomology of  $\mathcal{F}_x$  is canonically isomorphic to the  $\ell$ -adic nearby cohomology of  $f$  at  $x$  [6, 3.5], and the local motivic zeta function of  $f$  at  $x$  can be realized as a “Weil generating series” of  $\mathcal{F}_x$  [19, 9.7]. If  $X$  is normal at  $x$ , then  $\mathcal{F}_x$  is a complete invariant of the formal germ  $(f, x)$  [19, 8.8].

In the present article, we study the topology of  $\mathcal{F}_x$ , considered as a  $k((t))$ -analytic space in the sense of Berkovich [4]. We denote by  $\widehat{k((t))}^a$  the completion of an algebraic closure of  $k((t))$ . Our first main result (Theorem 4.10) describes the homotopy type of

$$\overline{\mathcal{F}}_x := \mathcal{F}_x \widehat{\times}_{k((t))} \widehat{k((t))}^a$$

in terms of a so-called strictly semi-stable reduction of the germ  $(f, x)$ . The second main result (Theorem 5.7) states that, if  $k = \mathbb{C}$ , the singular cohomology of  $\overline{\mathcal{F}}_x$  coincides with the weight zero part of the mixed Hodge structure on the nearby cohomology of  $f$  at  $x$  [25][17].

These are local analogs of results by Berkovich [3]. He showed that the weight zero part of the limit mixed Hodge structure of a proper family  $Y \rightarrow \operatorname{Spec} \mathbb{C}[t]$  coincides with the singular cohomology of the nearby fiber  $\widehat{Y}_\eta \times_{\mathbb{C}((t))} \widehat{\mathbb{C}((t))}^a$ , where  $\widehat{Y}$  denotes the  $t$ -adic completion of  $Y$  and  $\widehat{Y}_\eta$  its generic fiber. Our proofs closely

follow the ideas in [3], but we need additional work to pass from the global situation to our local one. The most important tool in our arguments is Berkovich's description of the homotopy type of the generic fiber of a poly-stable formal scheme [7].

The above theorems have natural motivic counterparts. We give an expression for Denef and Loeser's motivic Milnor fiber of  $f$  at  $x$  in terms of a strictly semi-stable model (Theorem 6.11). As we showed in [19], this motivic Milnor fiber can be realized as a "motivic volume" of the analytic Milnor fiber  $\mathcal{F}_x$ . The expression in Theorem 6.11 is similar in spirit to our description of the homotopy type of  $\mathcal{F}_x$  but uses different techniques (motivic integration) and yields another type of information (class in the Grothendieck ring).

Let us give a survey of the structure of the paper. In Section 2, we recall some basic notions about special formal schemes and their generic fibers, and about the analytic Milnor fiber. Section 3 contains the main technical tools of the paper: we define the simplicial set associated to a strictly semi-stable  $k$ -variety  $X$ , and we explain how it can be used to describe the homotopy type of the generic fiber of a strictly semi-stable formal scheme  $\mathcal{X}$ , following results by Berkovich. Moreover, we prove a crucial result about the homotopy type of the generic fiber of the completion of  $X$  along a union of irreducible components (Proposition 3.7 and its Corollary 3.8). It is this property that allows to pass from Berkovich's global situation to our local one.

In Section 4, these tools are applied to establish homotopy-equivalences between certain analytic spaces, and to study the homotopy type of the analytic Milnor fiber (Theorem 4.10). The key result is Proposition 4.4, which we now briefly explain. Any special formal  $k[[t]]$ -scheme  $\mathcal{X}$  can be considered as a special formal  $k$ -scheme  $\mathcal{X}^k$ , by forgetting the  $k[[t]]$ -structure. The generic fiber of  $\mathcal{X}^k$  (where  $k$  carries the trivial absolute value) is naturally fibered over  $[0, 1[$  by evaluating the points of the generic fiber (which are multiplicative semi-norms) in  $t$ . Proposition 4.4 studies the homotopy type of the fibers of this family.

Section 5 contains the results concerning the mixed Hodge structure on the nearby cohomology of  $f$  at  $x$  (where  $k = \mathbb{C}$ ). We recall Peters and Steenbrink's construction of this mixed Hodge structure, and we show how the weight zero part can be computed on a strictly semi-stable reduction (Proposition 5.6). Combining this result with the ones in Section 4, we see that the singular cohomology of the analytic Milnor fiber coincides with the weight zero part of the mixed Hodge structure on the nearby cohomology of  $f$  at  $x$  (Theorem 5.7).

In the final Section 6, we compare the preceding results with the motivic setting, and give an expression for Denef and Loeser's motivic nearby cycles and motivic Milnor fiber in terms of a strictly semi-stable model (Theorem 6.11 and Corollary 6.12).

## 2. PRELIMINARIES

**2.1. Some notation.** If  $S$  is any scheme, a  $S$ -variety is a separated reduced scheme of finite type over  $S$ .

For any locally ringed space  $Y$ , we denote by  $|Y|$  its underlying topological space; if no risk of confusion arises, we'll omit the vertical bars and we'll denote the underlying topological space by  $Y$ , to simplify notation.

For any field  $F$ , we denote by  $F^a$  an algebraic closure, and by  $F^s$  the separable closure of  $F$  in  $F^a$ . If  $L$  is a non-archimedean field (we do not exclude the trivial valuation), then the absolute value on  $L$  extends uniquely to an absolute value on  $L^a$ , and we denote by  $\widehat{L^s}$  and  $\widehat{L^a}$  the completions of  $L^s$ , resp.  $L^a$  (these completions coincide if the valuation is non-trivial). We will work in the category of  $L$ -analytic spaces as introduced by Berkovich [5]. For any  $L$ -analytic space  $X$ , we put  $\overline{X} = X \widehat{\times}_L \widehat{L^a}$ .

If  $R$  is a discrete valuation ring, with residue field  $k$ , and  $Y$  is a scheme over  $R$ , then we denote its special fiber  $Y \times_R k$  by  $Y_s$ .

If  $T$  is any topological space and  $S$  is a subspace of  $T$ , then a strong deformation retract of  $T$  onto  $S$  is a continuous map  $\phi : T \times [0, 1] \rightarrow T$  such that  $\phi(\cdot, 0)$  is the identity,  $\phi(s, r) = s$  for any point  $s$  of  $S$  and any  $r \in [0, 1]$ , and  $\phi(\cdot, 1)$  is a surjection onto  $S$ .

**2.2. Strictly semi-stable (formal) schemes.** Let  $K$  be a complete discretely valued field, with valuation  $v_K : K^* \rightarrow \mathbb{Z}$ . We do not assume that the valuation is non-trivial, but we will assume that it is normalized (i.e.  $v_K(K^*) = \mathbb{Z}$  or  $v_K(K^*) = \{0\}$ ). We denote by  $K^o$  the ring of integers in  $K$ , and by  $\tilde{K}$  the residue field. If  $v_K$  is trivial, then  $K = K^o = \tilde{K}$ . For each  $r \in ]0, 1[$ , we define an absolute value  $|\cdot|_r$  on  $K$  by  $|x|_r = r^{v_K(x)}$  and we denote by  $K_r = (K, |\cdot|_r)$  the corresponding non-archimedean field. Moreover, we put  $K_0 = (\tilde{K}, |\cdot|_0)$  where  $|\cdot|_0$  is the trivial absolute value. The distinction between  $K$  (which only carries a discrete valuation) and the fields  $K_r$  (which are non-archimedean fields) is crucial for the applications in this article.

If  $L = (L, |\cdot|_L)$  is any non-archimedean field, then we denote by  $L^o$  the ring of integers and by  $\tilde{L}$  the residue field. In particular,  $K_r^o = K^o$  and  $\tilde{K}_r = \tilde{K}$  for  $r \in ]0, 1[$ . A formal scheme  $\mathcal{X}$  over  $L^o$  is called *stft* if it is separated, and topologically of finite type over  $L^o$ . We denote by  $\mathcal{X}_\eta$  its generic fiber, a  $L$ -analytic space, and by  $\mathcal{X}_s$  its special fiber, a separated scheme of finite type over  $\tilde{L}$ . There is a natural specialization map (of sets<sup>1</sup>)  $sp_{\mathcal{X}} : \mathcal{X}_\eta \rightarrow \mathcal{X}_s$ . If  $X$  is a separated scheme of finite type over  $L$ , we denote by  $X^{an}$  the  $L$ -analytic space associated to  $X$  via non-archimedean GAGA [4, 3.4-5].

A special formal  $K^o$ -scheme is a separated adic Noetherian formal scheme  $\mathcal{X}$ , endowed with a morphism  $\mathcal{X} \rightarrow \mathrm{Spf} K^o$ , such that  $\mathcal{X}/\mathcal{J}$  is of finite type over  $K^o$  for any ideal of definition  $\mathcal{J}$  on  $\mathcal{X}$  (this definition is slightly more restrictive than the one used by Berkovich in [6]). In particular, any *stft* formal  $K^o$ -scheme is special. We denote by  $(SpF/K^o)$  the category of special formal  $K^o$ -schemes.

There is a functor

$$(\cdot)_0 : (SpF/K^o) \rightarrow (sft/\tilde{K}) : \mathcal{X} \mapsto \mathcal{X}_0$$

to the category  $(sft/\tilde{K})$  of separated  $\tilde{K}$ -schemes of finite type, where  $\mathcal{X}_0$  is the *reduction* of  $\mathcal{X}$  (the closed subscheme of  $\mathcal{X}$  defined by the largest ideal of definition in  $\mathcal{O}_{\mathcal{X}}$ ).

Moreover, we consider the functor

$$(\cdot)_s : (SpF/K^o) \rightarrow (SpF/\tilde{K}) : \mathcal{X} \mapsto \mathcal{X}_s = \mathcal{X} \times_{\mathrm{Spf} K^o} \mathrm{Spec} \tilde{K}$$

<sup>1</sup>In fact, if  $|\cdot|_L$  is non-trivial,  $sp_{\mathcal{X}}$  can be enhanced to a morphism of ringed sites: the specialization morphism  $sp_{\mathcal{X}} : \mathcal{X}_\eta \rightarrow \mathcal{X}_s$ , where  $\mathcal{X}_\eta$  is endowed with the strong  $G$ -topology; note that  $|\mathcal{X}| = |\mathcal{X}_s|$ .

mapping  $\tilde{\mathcal{X}}$  to its special fiber  $\mathcal{X}_s$ . If  $\mathcal{X}$  is *stft* over  $K^o$ , then  $\mathcal{X}_s$  is a scheme, and  $\mathcal{X}_0$  is the maximal reduced closed subscheme of  $\mathcal{X}_s$ ; in any case,  $(\mathcal{X}_s)_0 \cong \mathcal{X}_0$ .

For any special formal  $K^o$ -scheme  $\mathcal{X}$  and any value  $r \in ]0, 1[$ , we denote by  $\mathcal{X}(r)$  the formal scheme  $\mathcal{X}$  viewed as a special formal  $K_r^o$ -scheme via the identification  $K^o = K_r^o$ . We denote by  $\mathcal{X}(r)_\eta$  the generic fiber of  $\mathcal{X}(r)$  in the category of  $K_r$ -analytic spaces. We also put  $\mathcal{X}(0) = \mathcal{X}_s$ . It is a special formal scheme over  $\tilde{K}$ , and we denote its generic fiber in the category of  $K_0$ -analytic spaces by  $\mathcal{X}(0)_\eta$ . For each  $r \in [0, 1[$ , there is a canonical specialization map (of sets)  $sp_{\mathcal{X}(r)} : \mathcal{X}(r)_\eta \rightarrow \mathcal{X}_0$ .

If  $X$  is a separated  $\tilde{K}$ -scheme of finite type, then we can view  $X$  also as a *stft* formal  $\tilde{K}$ -scheme, and there exists a canonical morphism of  $K_0$ -analytic spaces  $X_\eta \rightarrow X^{an}$ , which is an isomorphism iff  $X$  is proper [27, 1.10]. Hence, if  $\mathcal{X}$  is a proper *stft* formal  $R$ -scheme, then  $\mathcal{X}(0)_\eta$  is canonically isomorphic to  $\mathcal{X}_s^{an}$ .

A *stft* formal  $K^o$ -scheme is called strictly semi-stable if it can be covered with affine open formal subschemes  $\mathcal{U}$ , endowed with an étale morphism of formal  $K^o$ -schemes of the form

$$\mathcal{U} \rightarrow \mathrm{Spf} K^o\{x_0, \dots, x_m\}/(x_0 \cdot x_1 \cdot \dots \cdot x_p - \pi)$$

for some  $p \leq m$ , where  $\pi$  generates the maximal ideal of  $K^o$  (in particular,  $\pi = 0$  if the valuation on  $K$  is trivial).

This definition includes as a special case the class of strictly semi-stable  $\tilde{K}$ -varieties, i.e. varieties which admit Zariski-locally an étale map to a  $\tilde{K}$ -scheme of the form  $\mathrm{Spec} \tilde{K}[x_0, \dots, x_m]/(x_0 \cdot x_1 \cdot \dots \cdot x_p)$ . If  $\mathcal{X}$  is a strictly semi-stable formal  $K^o$ -scheme, then  $\mathcal{X}_s$  is a strictly semi-stable  $\tilde{K}$ -variety.

If  $X$  is a strictly semi-stable  $\tilde{K}$ -variety, we denote by  $\mathrm{Irr}(X)$  its set of irreducible components. For any non-empty subset  $J$  of  $\mathrm{Irr}(X)$ , we put  $X_J = \bigcap_{V \in J} V$  and  $X_J^o = X_J \setminus \bigcup_{W \in (\mathrm{Irr}(X) \setminus J)} W$ .

**2.3. The analytic Milnor fiber.** Let  $k$  be any field, and put  $K = k((t))$ , endowed with the  $t$ -adic valuation. Fix a value  $r \in ]0, 1[$ . Let  $X$  be a variety over  $k$ , endowed with a morphism of  $k$ -schemes  $f : X \rightarrow \mathrm{Spec} k[t]$ . The  $t$ -adic completion of  $f$  is a *stft* formal  $K^o$ -scheme  $\mathcal{X}$ , whose special fiber  $\mathcal{X}_s$  is canonically isomorphic to the fiber of  $f$  over the origin. For any closed point  $x$  of  $\mathcal{X}_s$ , the set  $sp_{\mathcal{X}(r)}^{-1}(x)$  is open in  $\mathcal{X}(r)_\eta$  and inherits the structure of a  $K_r$ -analytic space. By [9, 0.2.7] it is canonically isomorphic to the generic fiber of the special formal  $K_r^o$ -scheme  $\mathrm{Spf} \hat{\mathcal{O}}_{X,x}$ , where the  $K_r^o$ -structure is defined by  $f$ . In [21], we called  $sp_{\mathcal{X}(r)}^{-1}(x)$  the *analytic Milnor fiber* of  $f$  at  $x$ , and we denoted it by  $\mathcal{F}_x$ . By [6, 3.5], if  $k$  is separably closed, the  $\ell$ -adic cohomology of  $\mathcal{F}_x$  is canonically isomorphic to the cohomology of the stalk of the complex of  $\ell$ -adic nearby cycles at  $x$  (here  $\ell$  is a prime invertible in  $k$ ): for each integer  $i \geq 0$ , there is a canonical  $G(K^s/K)$ -equivariant isomorphism

$$H_{\acute{e}t}^i(\mathcal{F}_x \hat{\times}_{K_r} \widehat{K_r^s}, \mathbb{Q}_\ell) \cong R^i \psi_\eta(\mathbb{Q}_\ell)_x$$

### 3. THE SIMPLICIAL SET ASSOCIATED TO A STRICTLY SEMI-STABLE VARIETY

**3.1. Definitions.** In this section, we define the simplicial complex  $\Delta(X)$  associated to a strictly semi-stable  $\tilde{K}$ -variety  $X$ , and we list some basic properties.

**Remark.** In [27, § 4.1], Thuillier defines  $\Delta(X)$  as an oriented simplicial complex by choosing a total order on the set of irreducible components of  $X$ . It seems more natural to construct  $\Delta(X)$  as an unoriented simplicial set, independent of

all choices. This is the approach we adopt here. The result is isomorphic to the unoriented simplicial set underlying Thuillier's construction.  $\square$

For any pair of categories  $\mathcal{C}, \mathcal{D}$ , we denote by  $\mathcal{D}^o\mathcal{C}$  the category of presheaves on  $\mathcal{D}$  with values in  $\mathcal{C}$ . Its objects are functors  $\mathcal{D}^o \rightarrow \mathcal{C}$ , where  $\mathcal{D}^o$  denotes the opposite category of  $\mathcal{D}$ , and its morphisms are natural transformations of functors. We denote by  $(Ens)$  the category of sets.

For any integer  $n \geq 0$ , we denote by  $[n]$  the set  $\{0, \dots, n\}$ , and we define a category  $\Delta$  whose objects are the sets  $[n]$  for  $n \geq 0$ , and whose arrows are maps of sets. The category  $\Delta^o(Ens)$  is called the category of (unoriented) simplicial sets.

The object of  $\Delta^o(Ens)$  represented by  $[n]$  is called the standard  $n$ -simplex, and denoted by  $\Delta[n]$ . If  $\Sigma$  is a simplicial set and  $n \geq 0$  is an integer, we call  $\Sigma([n])$  the set of  $n$ -simplices of  $\Sigma$ . Note that there exists a natural bijection  $\Sigma([n]) \cong Hom_{\Delta^o(Ens)}(\Delta[n], \Sigma)$ . A  $n$ -simplex  $\gamma$  is called degenerate if there exists a map of sets  $g : [n] \rightarrow [n-1]$  such that  $\gamma$  belongs to the image of  $\Sigma(g) : \Sigma([n-1]) \rightarrow \Sigma([n])$ . A  $m$ -simplex  $\gamma_1$  and a  $n$ -simplex  $\gamma_2$  are called equivalent if there exist maps  $[m] \rightarrow [n]$  and  $[n] \rightarrow [m]$  mapping  $\gamma_2$  to  $\gamma_1$ , resp.  $\gamma_1$  to  $\gamma_2$ . For any set  $I$ , we define the associated simplicial set  $\Delta_I$  by

$$\Delta_I([n]) = Hom_{(Ens)}([n], I)$$

for all  $n \geq 0$ .

There is a natural geometric realization functor

$$|\cdot| : \Delta^o(Ens) \rightarrow (Ke) : \Sigma \mapsto |\Sigma|$$

where  $(Ke)$  denotes the category of Kelley spaces (topological Hausdorff spaces  $T$  such that a subset of  $T$  is closed iff its intersection with all compact subsets of  $T$  is closed). This functor is characterized by the fact that it commutes with direct limits, maps the standard  $n$ -simplex  $\Delta[n]$  to the topological  $n$ -simplex

$$\Delta_n = \{(u_0, \dots, u_n) \in [0, 1]^{n+1} \mid \sum_{i=0}^n u_i = 1\}$$

and sends a map of sets  $\alpha : [m] \rightarrow [n]$  to the unique affine map  $|\alpha| : \Delta_m \rightarrow \Delta_n$  sending the vertex  $v_i$  of  $\Delta_m$  to the vertex  $v_{\alpha(i)}$  of  $\Delta_n$  (for  $n \geq 0$  and  $i \in [n]$ , the vertex  $v_i$  of  $\Delta_n$  is the point with coordinates  $u_j = \delta_{ij}$ ,  $j = 0, \dots, n$ ).

If  $\Sigma$  is a simplicial set, then a  $n$ -cell of  $|\Sigma|$  is the image of the interior  $(\Delta_n)^o$  under the map  $|\gamma| : \Delta_n \rightarrow |\Sigma|$  induced by some non-degenerate  $n$ -simplex  $\gamma \in \Sigma([n])$ . Two simplices  $\gamma_1 \in \Sigma([m])$  and  $\gamma_2 \in \Sigma([n])$  are equivalent, iff the images of the corresponding maps  $|\gamma_1| : \Delta_m \rightarrow |\Sigma|$  and  $|\gamma_2| : \Delta_n \rightarrow |\Sigma|$  coincide.

Now let  $X$  be a strictly semi-stable  $\tilde{K}$ -variety. We denote by  $Str(X)$  the set consisting of the generic points of the closed subsets  $X_J$  of  $X$ , where  $J$  varies over the non-empty subsets of  $Irr(X)$ . In particular,  $Str(X_J) \subset Str(X)$  is the set of generic points of the smooth variety  $X_J$ . Identifying an element of  $Irr(X)$  with its generic point, we can consider  $Irr(X)$  as a subset of  $Str(X)$  in a natural way.

For any point  $x$  of  $X$ , we denote by  $\Psi(x)$  the set of elements of  $Irr(X)$  containing  $x$ ; if we want to make  $X$  explicit, we write  $\Psi_X(x)$  instead of  $\Psi(x)$ . For any point  $x$  of  $Str(X)$ , we denote by  $S_x$  the connected component of  $X_{\Psi(x)}^o$  containing  $x$ . Then  $x$  is the generic point of  $S_x$ , and  $\{S_x \mid x \in Str(X)\}$  is a finite stratification of  $X$  into smooth irreducible locally closed subsets. We call any union of strata a strata subset of  $X$ .

We define a partial ordering on  $Str(X)$  as follows:  $x \leq y$  iff  $y$  belongs to the Zariski closure of  $\{x\}$  in  $X$ . For each  $x \in Str(X)$  we consider the simplicial set  $\Delta(x) = \Delta_{\Psi(x)}$ . For  $x \leq y$ , we have  $\Psi(x) \subset \Psi(y)$ , and hence a natural morphism of simplicial sets  $\Delta(x) \rightarrow \Delta(y)$ . This defines a functor

$$\Delta(\cdot) : Str(X) \rightarrow \Delta^o(Ens)$$

and we define  $\Delta(X)$  as its colimit:

$$\Delta(X) = \varinjlim_{Str(X)} \Delta(\cdot)$$

We can give a more explicit construction of  $\Delta(X)$  as follows. Denote, for each  $n \geq 0$ , by  $D(X)([n])$  the set of couples  $(x, f)$  with  $x \in Str(X)$ , and  $f$  a surjection  $[n] \rightarrow \Psi(x)$ . For any map  $\alpha : [m] \rightarrow [n]$ , we define a map

$$D(X)(\alpha) : D(X)([n]) \rightarrow D(X)([m]) : (x, f) \mapsto (x', f')$$

where  $f' := f \circ \alpha$ , and  $x'$  is the unique point of  $Str(X)$  such that  $\Psi(x') = Im(f')$  and  $x$  belongs to the Zariski closure of  $\{x'\}$  in  $X$ . In this way,  $D(X)$  becomes a simplicial set. The following lemma can be verified in a straightforward way.

**Lemma 3.1.** *There exists a canonical isomorphism of simplicial sets  $\Delta(X) \cong D(X)$ . In particular, for each  $n \geq 0$ , there is a canonical bijection between the set of equivalence classes of non-degenerate  $n$ -simplices (or, equivalently, the set of  $n$ -cells) of  $\Delta(X)$  and the set  $\cup_{J \subset Irr(X), |J|=n+1} Str(X_J)$ .*

Hence, the non-degenerate  $n$ -simplices  $\gamma$  of  $\Delta(X)$  correspond to couples  $(x, f)$  with  $x \in Str(X)$  and  $f$  a bijection  $[n] \cong \Psi(x)$ . A point  $z$  of the corresponding cell of  $|\Delta(X)|$  is the image of a unique point  $(u_0, \dots, u_n)$  of  $(\Delta_n)^o$  under the map  $|\gamma| : \Delta_n \rightarrow |\Delta(X)|$ , and we define a tuple  $v \in ]0, 1]^{\Psi(x)}$  by  $v(i) = u_{f^{-1}(i)}$  for  $i \in \Psi(x)$ . This tuple is invariant under equivalence of non-degenerate  $n$ -simplices, and the point  $z$  of  $|\Delta(X)|$  is completely determined by the couple  $(x, v)$ .

**Definition 3.2** (Barycentric representation). *We call  $v$  the tuple of barycentric coordinates of the point  $z$ , and we call  $(x, v)$  the barycentric representation of  $z$ .*

By Lemma 3.1, there is a natural bijection  $x \mapsto C_x$  from  $Str(X)$  to the set of cells of  $|\Delta(X)|$ . Whenever  $E$  is a strata subset of  $X$ , we denote by  $|\Delta_E(X)|$  the union of the cells  $C_x$  with  $x \in Str(X) \cap E$ .

**3.2. Comparison with Berkovich' definition.** Denote by  $\tilde{\Delta}$  the subcategory of  $\Delta$  with the same objects but with only injective maps. This is a full subcategory of the category  $\Lambda$  considered in [7, p. 24]. If  $X$  is a strictly semi-stable  $\tilde{K}$ -variety, then Berkovich defines in [7, p. 29] an object  $C(X)$  of the category  $\Lambda^o(Ens)$  of presheaves on  $\Lambda$ , and its geometric realization  $|C(X)|$ . The aim of this section is to compare these objects with the simplicial set  $\Delta(X)$  defined above, and its geometric realization  $|\Delta(X)|$ .

A simplicial set  $\Sigma$  is called non-degenerate, if for any injective map  $[m] \rightarrow [n]$  in  $\Delta$ , the induced map  $\Sigma([n]) \rightarrow \Sigma([m])$  takes non-degenerate  $n$ -simplices to non-degenerate  $m$ -simplices. A morphism of simplicial sets is called non-degenerate if it takes non-degenerate simplices to non-degenerate ones. We denote the subcategory of  $\Delta^o(Ens)$  consisting of non-degenerate simplicial sets with non-degenerate morphisms between them by  $\Delta^o(Ens)_{nd}$ .

The embedding of  $\tilde{\Delta}$  in  $\Lambda$ , resp.  $\Delta$  induces forgetful functors  $F : \Lambda^o(Ens) \rightarrow (\tilde{\Delta})^o(Ens)$  and  $G : \Delta^o(Ens) \rightarrow (\tilde{\Delta})^o(Ens)$ .

**Lemma 3.3.** *The functor  $G : \Delta^o(Ens) \rightarrow (\tilde{\Delta})^o(Ens)$  has a left adjoint  $H : (\tilde{\Delta})^o(Ens) \rightarrow \Delta^o(Ens)$ . This functor is a faithful embedding, its image is contained in  $\Delta^o(Ens)_{nd}$ , and  $H : (\tilde{\Delta})^o(Ens) \rightarrow \Delta^o(Ens)_{nd}$  is an equivalence of categories.*

*Proof.* For any integer  $n \geq 0$ , we denote by  $\tilde{\Delta}[n]$  the object of  $(\tilde{\Delta})^o(Ens)$  represented by  $[n]$ . For any object  $S$  of  $(\tilde{\Delta})^o(Ens)$ , we denote by  $\tilde{\Delta}/S$  the category of morphisms  $\alpha : \tilde{\Delta}[n] \rightarrow S$ , with  $n \geq 0$ , where a morphism in  $\tilde{\Delta}/S$  from  $\alpha$  to  $\beta : \tilde{\Delta}[m] \rightarrow S$  is a map  $f : \tilde{\Delta}[n] \rightarrow \tilde{\Delta}[m]$  in  $(\tilde{\Delta})^o(Ens)$  with  $\alpha = \beta \circ f$ . We define the functor  $H$  by putting  $H_S(\alpha) = \Delta[n]$  for any object  $\alpha : \tilde{\Delta}[n] \rightarrow S$  of  $\tilde{\Delta}/S$ , and

$$H(S) = \varinjlim_{\tilde{\Delta}/S} H_S(\cdot)$$

In particular,  $H(\tilde{\Delta}[n]) = \Delta[n]$ . The action of  $H$  on morphisms in  $(\tilde{\Delta})^o(Ens)$  is the obvious one.

Now we check that  $H$  is indeed a left adjoint for  $G$ . Let  $\Sigma$  be any object of  $\Delta^o(Ens)$ , and let  $S$  be any object of  $(\tilde{\Delta})^o(Ens)$ . By definition,

$$Hom_{\Delta^o(Ens)}(H(S), \Sigma) = \varinjlim_{\tilde{\Delta}[n] \rightarrow S} \Sigma([n])$$

and since

$$\Sigma([n]) = G(\Sigma)([n]) = Hom_{(\tilde{\Delta})^o(Ens)}(\tilde{\Delta}[n], G(\Sigma))$$

it suffices to note that

$$S \cong \varinjlim_{\tilde{\Delta}[n] \rightarrow S} \tilde{\Delta}[n]$$

for any object  $S$  of  $(\tilde{\Delta})^o(Ens)$ , by [15, II.1.1].

For any object  $S$  of  $(\tilde{\Delta})^o(Ens)$ , and each integer  $n \geq 0$ , we consider the set  $\mathcal{S}_n(S)$  consisting of triples  $(p, f, \gamma)$  where  $p \geq 0$ ,  $f$  is a surjection  $[n] \rightarrow [p]$ , and  $\gamma$  is an element of  $S([p])$ . We define an equivalence relation on  $\mathcal{S}_n(S)$  by stipulating that  $(p, f, \gamma) \sim (p', f', \gamma')$  iff  $p = p'$  and there exists an automorphism  $\varphi$  of  $[p]$  such that  $f' = \varphi \circ f$  and  $\gamma = S(\varphi)(\gamma')$ .

It is not hard to see that, for each integer  $n \geq 0$ , there exists a canonical bijection between the set of  $n$ -simplices  $H(S)([n])$  and the quotient set  $\mathcal{S}_n(S)/\sim$ . The  $n$ -simplex represented by  $(p, f, \gamma)$  is non-degenerate iff  $p = n$ . If  $\alpha : [m] \rightarrow [n]$  is a morphism in  $\Delta$ , then

$$H(S)(\alpha) : H(S)([n]) \rightarrow H(S)([m])$$

maps  $(p, f, \gamma)$  to

$$(q, (f \circ \alpha : [m] \rightarrow Im(f \circ \alpha) \cong [q]), \gamma')$$

where we chose an isomorphism  $[q] \cong Im(f \circ \alpha)$  and  $\gamma'$  is the image of  $\gamma$  in  $H(S)([q])$  w.r.t. the inclusion map  $[q] \cong Im(f \circ \alpha) \rightarrow [p]$ . Using this description, one can see that for any morphism  $h : S \rightarrow T$  in  $(\tilde{\Delta})^o(Ens)$ , the image  $H(h) : H(S) \rightarrow H(T)$  is a non-degenerate morphism between non-degenerate simplicial sets.

To conclude, we construct a quasi-inverse  $I$  for  $H : (\tilde{\Delta})^o(Ens) \rightarrow \Delta^o(Ens)_{nd}$ . For any non-degenerate simplicial set  $\Sigma$  and any  $n \geq 0$ , we define  $I(\Sigma)([n])$  as the set of non-degenerate  $n$ -simplices of  $\Sigma$ . By the definitions of non-degenerate

simplicial set and non-degenerate morphism, we can make  $I$  into a functor in the obvious way. It is easy to see that  $I$  is quasi-inverse to  $H$ .  $\square$

**Proposition 3.4.** *For any strictly semi-stable  $\tilde{K}$ -variety  $X$ , there exists a canonical isomorphism of simplicial sets  $\alpha : (H \circ F)(C(X)) \rightarrow \Delta(X)$ . In particular,  $\Delta(X)$  is non-degenerate. Moreover, there exists a canonical homeomorphism  $|\Delta(X)| \rightarrow |C(X)|$ .*

*Proof.* Recall that, for any point  $x$  of  $Str(X)$ , we denote by  $\Psi(x)$  the set of irreducible components of  $X$  containing  $x$ . Then, by definition, for any integer  $n \geq 0$ ,  $F(C(X))([n])$  is the set of pairs  $(x, \alpha)$  with  $x \in Str(X)$  and  $\alpha$  a bijection  $[n] \rightarrow \Psi(x)$ . By the construction of the functor  $H$  in the proof of Lemma 3.3, we see that  $(H \circ F)(C(X))([n])$  is the set of pairs  $(x, \beta)$  with  $x \in Str(X)$  and  $\beta$  a surjection  $[n] \rightarrow \Psi(x)$ . Now the existence of the canonical isomorphism follows from Lemma 3.1. The existence of a canonical homeomorphism  $|(H \circ F)(C(X))| \rightarrow |C(X)|$  is clear from Berkovich' construction of  $|C(X)|$ .  $\square$

**3.3. The skeleton of a strictly semi-stable formal scheme.** Let  $(L, |\cdot|_L)$  be an arbitrary non-archimedean field, with ring of integers  $L^\circ$  and residue field  $\tilde{L}$  (we do not exclude the trivial absolute value). We recall a particular case of Berkovich' definition of the skeleton  $S(\mathcal{X})$  of a poly-stable formal  $L^\circ$ -scheme  $\mathcal{X}$ , and his construction of a strong deformation retract of  $\mathcal{X}_\eta$  onto  $S(\mathcal{X})$  (see [7, § 5]). We establish some basic properties for use in the following sections.

*Case 1.* Suppose that  $\mathcal{X} = \text{Spf } A$  with  $A = L^\circ\{x_0, \dots, x_m\}/(x_0 \cdot \dots \cdot x_p - \alpha)$  for some  $p \leq m$ , and with  $\alpha \in L^\circ$ ,  $|\alpha|_L < 1$ . Each element of  $A_L := A \otimes_{L^\circ} L$  has a unique representant in the set

$$D = \left\{ \sum_{i \in \mathbb{N}^{[m]}} a_i x^i \in L\{x_0, \dots, x_m\} \mid a_i = 0 \text{ if } \min\{i_0, \dots, i_p\} > 0 \right\}$$

i.e. the natural map  $D \rightarrow A_L$  is a bijection (and even an isometry if we endow  $A_L$  with the quotient norm w.r.t. the given presentation  $L\{x_0, \dots, x_m\} \rightarrow A_L$ ). Put

$$S = \{r \in [0, 1]^{[p]} \mid r_0 \cdot \dots \cdot r_p = |\alpha|_L\}$$

and consider the map  $\theta : S \rightarrow \mathcal{X}_\eta$  mapping  $r$  to the element  $\theta(r)$  of  $\mathcal{X}_\eta = \mathcal{M}(A_L)$  defined by

$$\theta(r) : D \rightarrow \mathbb{R}_+ : \sum_i a_i x^i \mapsto \max_{i \in \mathbb{N}^{[m]}} \{|a_i|_L r^i\}$$

where  $r^i = r_0^{i_0} \cdot \dots \cdot r_p^{i_p}$  with the convention that  $0^0 = 1$ . The map  $\theta$  is a homeomorphism onto its image  $\theta(S)$ , which is by definition the skeleton  $S(\mathcal{X})$  of  $\mathcal{X}$ , and  $\theta$  is right inverse to the map  $\phi : \mathcal{X}_\eta \rightarrow S$  mapping a point  $z$  of  $\mathcal{X}_\eta$  to the tuple  $(|x_0(z)|, \dots, |x_p(z)|)$ . If we put  $\tau_{\mathcal{X}} = \theta \circ \phi$ , then Berkovich constructed a natural strong deformation retract  $\Phi_{\mathcal{X}} : \mathcal{X}_\eta \times [0, 1] \rightarrow \mathcal{X}_\eta$  with  $\Phi_{\mathcal{X}}(\cdot, 1) = \tau_{\mathcal{X}}$ .

*Case 2.* Now we consider the case where  $\mathcal{X}$  admits an étale map  $h : \mathcal{X} \rightarrow \mathcal{Y} = \text{Spf } A$ , with  $A$  as above. Then the skeleton  $S(\mathcal{X})$  is the inverse image of  $S(\mathcal{Y})$  under  $h_\eta$ , and it does not depend on the choice of the map  $h$ . Moreover, Berkovich describes the map  $\Phi_{\mathcal{Y}}$  in terms of a certain torus action, which lifts uniquely to  $\mathcal{X}_\eta$ ; in this way, he defines a natural strong deformation retract  $\Phi_{\mathcal{X}} : \mathcal{X}_\eta \times [0, 1] \rightarrow \mathcal{X}_\eta$  of  $\mathcal{X}_\eta$  onto  $S(\mathcal{X})$ , such that  $\Phi_{\mathcal{X}}(h_\eta(x), \rho) = h_\eta \circ \Phi_{\mathcal{Y}}(x, \rho)$  for each point  $x$  of  $\mathcal{X}_\eta$  and each  $\rho \in [0, 1]$ . This map  $\Phi_{\mathcal{X}}$  does not depend on  $h$ .



*Case 3.* Finally, if  $\mathcal{X}$  is a *stft* formal  $L^o$ -scheme that can be covered by open formal subschemes with the property of Case 2, then the construction of the skeleton  $S(\mathcal{X})$  and the strong deformation retract  $\Phi_{\mathcal{X}}$  are obtained by gluing the constructions in the previous step. We put  $\tau_{\mathcal{X}} = \Phi_{\mathcal{X}}(\cdot, 1) : \mathcal{X}_{\eta} \rightarrow S(\mathcal{X})$ . In particular, Case 3 applies to all strictly semi-stable formal  $K^o$ -schemes.

The map  $\Phi_{\mathcal{X}}$  has the following property: for any point  $x$  of  $\mathcal{X}_{\eta}$ , we have  $(sp_{\mathcal{X}} \circ \Phi_{\mathcal{X}})(x, \rho) = sp_{\mathcal{X}}(x)$  for  $\rho \in [0, 1]$ , and  $(sp_{\mathcal{X}} \circ \tau_{\mathcal{X}})(x)$  is the generic point of the stratum of the strictly semi-stable  $\tilde{L}$ -variety  $\mathcal{X}_s$  containing  $sp_{\mathcal{X}}(x)$ . In particular, if  $E$  is a strata subset of  $\mathcal{X}_s$ , then  $\Phi_{\mathcal{X}}$  restricts to a strong deformation retract

$$sp_{\mathcal{X}}^{-1}(E) \times [0, 1] \rightarrow sp_{\mathcal{X}}^{-1}(E)$$

onto  $S(\mathcal{X}) \cap sp_{\mathcal{X}}^{-1}(E)$ .

Berkovich constructed a natural homeomorphism  $S(\mathcal{X}) \cong |\Delta(\mathcal{X}_s)|$  (here we use the canonical homeomorphism  $|\Delta(\mathcal{X}_s)| \cong |C(\mathcal{X}_s)|$  established in Proposition 3.4). If  $E$  is a strata subset of  $\mathcal{X}_s$ , then this homeomorphism identifies  $S(\mathcal{X}) \cap sp_{\mathcal{X}}^{-1}(E)$  with  $|\Delta_E(\mathcal{X}_s)|$  (cf. [7, 5.4]).

We will give an explicit description of the composed map

$$\tau_{\mathcal{X}(r)} : \mathcal{X}(r)_{\eta} \rightarrow S(\mathcal{X}(r)) \cong |\Delta(\mathcal{X}_s)|$$

if  $\mathcal{X}$  is a strictly semi-stable formal  $K^o$ -scheme, and  $r \in [0, 1]$ .

For  $n \geq 0$  and  $q \in [0, 1]$ , put

$$\Sigma_q^n = \{(u_0, \dots, u_n) \in [0, 1]^{[n]} \mid \prod_{i \in [n]} u_i = q\}$$

In [7, §4], Berkovich constructs a natural homeomorphism  $\alpha : \Delta_n \rightarrow \Sigma_q^n$ . It identifies the face of  $\Delta_n$  corresponding to a non-empty subset  $S$  of  $[n]$ , with the subspace of  $\Sigma_q^n$  defined by  $u_i = 1$  for  $i \notin S$  (note that this subspace is homeomorphic to  $\Sigma_q^{|S|-1}$ ). If  $w$  is a point of  $\Delta_n$ , we call the tuple  $\alpha(w)$  in  $\Sigma_q^n$  the  $q$ -coloured coordinates of  $w$ .

For any non-empty finite set  $I$  and any  $q \in [0, 1]$ ,  $\alpha$  induces a map

$$\alpha : \{u \in [0, 1]^I \mid \sum_{i \in I} u(i) = 1\} \rightarrow \{u \in [0, 1]^I \mid \prod_{i \in I} u(i) = q\}$$

by choosing a bijection  $I \cong [n]$  for some  $n \geq 0$ ; the map is independent of this choice.

**Definition 3.5** ( $q$ -coloured representation). *We fix a value  $q \in [0, 1]$ . Let  $X$  be a strictly semi-stable  $\tilde{K}$ -variety, and let  $z$  be a point of  $|\Delta(X)|$  with barycentric representation  $(x, v)$  (see Definition 3.2). We define the  $q$ -coloured coordinates of  $z$  as the image of  $v$  under the map*

$$\alpha : \{u \in [0, 1]^{\Psi(x)} \mid \sum_{i \in I} u(i) = 1\} \rightarrow \{u \in [0, 1]^{\Psi(x)} \mid \prod_{i \in I} u(i) = q\}$$

and we call  $(x, \alpha(v))$  the  $q$ -coloured representation of  $z$ .

**Lemma 3.6.** *Let  $\mathcal{X}$  be a strictly semi-stable formal  $K^o$ -scheme, and fix a value  $r \in [0, 1]$ . Let  $z_0$  be any point of the special fiber  $\mathcal{X}_s$ , and let  $x$  be the unique point of  $Str(\mathcal{X}_s)$  such that  $z_0$  is contained in the stratum  $S_x$ .*

*For each element  $C$  of  $\Psi(x)$ , we choose a generator  $T_C$  of the kernel of the natural morphism  $\mathcal{O}_{\mathcal{X}, z_0} \rightarrow \mathcal{O}_{C, z_0}$ . Then the image of a point  $z \in sp_{\mathcal{X}(r)}^{-1}(z_0)$  under the*

retraction  $\tau_{\mathcal{X}(r)} : \mathcal{X}(r)_\eta \rightarrow S(\mathcal{X}(r)) \cong |\Delta(\mathcal{X}_s)|$  is the point with  $|\pi|_{K_r}$ -coloured representation  $(x, (|T_i(z)|)_{i \in \Psi(x)})$ .

Recall that  $\pi$  is a generator of the maximal ideal of  $K^\circ$ .

*Proof.* If  $\mathcal{X}$  is of the form  $\mathrm{Spf} A$  with  $A = K^\circ\{x_0, \dots, x_m\}/(x_0 \dots x_p - \pi)$ , then this follows immediately from the constructions in [7, § 5]. So let us assume that  $\mathcal{X}$  admits an étale morphism  $h : \mathcal{X} \rightarrow \mathcal{Y} = \mathrm{Spf} A$ . Shrinking  $\mathcal{X}$  around  $z_0$ , we may assume that  $x$  is the unique maximal element of  $\mathrm{Str}(\mathcal{X}_s)$  (w.r.t. the partial order defined in Section 3.1), and that  $h$  induces a bijection  $\mathrm{Irr}(\mathcal{X}_s) \cong \mathrm{Irr}(\mathcal{Y}_s)$ . In this case,  $h$  induces isomorphisms  $\beta : \Delta(\mathcal{X}_s) \cong \Delta(\mathcal{Y}_s) = \Delta[p]$  and  $\gamma : S(\mathcal{X}(r)) \cong S(\mathcal{Y}(r))$ , by Step 6 in the proof of Theorems 5.2-4 in [7].

Since  $|\varphi(z)| = 1$  for any unit  $\varphi$  on  $\mathcal{X}$ , the value  $|T_i(z)|$  only depends on  $i$  and  $z$  and not on the choice of the generator  $T_i$ . Hence, we might as well take  $T_i = h^*x_i$  for  $i = 0, \dots, p$  (we used the bijection  $\mathrm{Irr}(\mathcal{X}_s) \cong \mathrm{Irr}(\mathcal{Y}_s)$  to identify  $\Psi(x)$  with  $\{0, \dots, p\}$ ). Therefore, it only remains to observe that the diagram

$$\begin{array}{ccc} |\Delta(\mathcal{X}_s)| & \longrightarrow & S(\mathcal{X}(r)) \\ \beta \downarrow & & \downarrow \gamma \\ |\Delta(\mathcal{Y}_s)| & \longrightarrow & S(\mathcal{Y}(r)) \end{array}$$

commutes, where the horizontal arrows are the natural homeomorphisms constructed by Berkovich (in fact, this is the definition of the upper horizontal arrow in Step 6 of Berkovich' proof [7, p. 48]).  $\square$

**3.4. Restriction to irreducible components.** Let  $X$  be a strictly semi-stable  $\tilde{K}$ -variety, and let  $E$  be a union of irreducible components of  $X$ . Of course,  $E$  itself is again a strictly semi-stable  $\tilde{K}$ -variety, and hence, it defines an associated simplicial set  $\Delta(E)$  with geometric realization  $|\Delta(E)|$ . In general,  $|\Delta(E)|$  is not homeomorphic to  $|\Delta_E(X)|$ . For instance, when  $X = \mathrm{Spec} \tilde{K}[x, y]/(xy)$  and  $E$  is the component  $x = 0$ , then  $|\Delta(E)|$  is a point while  $|\Delta_E(X)|$  is homeomorphic to the semi-open interval  $[0, 1[$ . However, it is clear from the definitions that the inclusion  $\mathrm{Str}(E) \subset \mathrm{Str}(X)$  induces an injective morphism of simplicial sets  $\Delta(E) \rightarrow \Delta(X)$  and a natural continuous injection  $|\Delta(E)| \rightarrow |\Delta_E(X)|$  which is a homeomorphism onto its image.

Denote by  $\mathfrak{E}$  the formal completion of  $X$  along  $E$ ; this is a special formal  $\tilde{K}$ -scheme. The closed immersion of special formal  $\tilde{K}$ -schemes  $h : E \rightarrow \mathfrak{E}$  induces a morphism of  $K_0$ -analytic spaces  $h_\eta : E_\eta \rightarrow \mathfrak{E}_\eta$ .

**Proposition 3.7.** *If  $X$  is a strictly semi-stable  $\tilde{K}$ -variety, and  $E$  is a union of irreducible components of  $X$ , then there exists a strong deformation retract*

$$\Phi_E^X : |\Delta_E(X)| \times [0, 1] \rightarrow |\Delta_E(X)|$$

of  $|\Delta_E(X)|$  onto  $|\Delta(E)|$  such that the diagram

$$\begin{array}{ccc} E_\eta & \xrightarrow{h_\eta} & \mathfrak{E}_\eta \\ \downarrow \tau_E & & \downarrow \tau_X \\ |\Delta(E)| & \xleftarrow{\Phi_E^X(\cdot, 1)} & |\Delta_E(X)| \end{array}$$

commutes.

*Proof.* We may assume  $E \neq X$ . Choose a sequence

$$Irr(E) = I(0) \subset I(1) \subset \dots \subset I(q+1) = Irr(X)$$

such that  $|I(i+1)| = |I(i)| + 1$  for  $i = 0, \dots, q$ , and put  $E^{(i)} = \cup_{V \in I(i)} V$ . Denote by  $\mathfrak{E}^{(i)}$  the formal completion of  $E^{(i)}$  along  $E$ , and by  $g^{(i)} : \mathfrak{E}^{(i)} \rightarrow \mathfrak{E}^{(i+1)}$  the natural closed immersion. It suffices to construct a strong deformation retract  $\Phi_{E^{(q)}}^X$  of  $|\Delta_E(X)|$  onto  $|\Delta_E(E^{(q)})|$  such that the diagram

$$(3.1) \quad \begin{array}{ccc} \mathfrak{E}_\eta^{(q)} & \xrightarrow{g_\eta^{(q)}} & \mathfrak{E}_\eta \\ \downarrow \tau_{E^{(q)}} & & \downarrow \tau_X \\ |\Delta_E(E^{(q)})| & \xleftarrow{\Phi_{E^{(q)}}^X(\cdot, 1)} & |\Delta_E(X)| \end{array}$$

commutes: iterating the construction, we get strong deformation retracts  $\Phi_{E^{(i)}}^{E^{(i+1)}}$  for  $i = 0, \dots, q$ , and these can be glued to obtain  $\Phi_E^X$ .

To simplify notation, we will denote  $E^{(q)}$  by  $Y$ ,  $\mathfrak{E}^{(q)}$  by  $\mathscr{Y}$ , and  $g^{(q)}$  by  $g$ . Denote the unique element in  $Irr(X) \setminus Irr(Y)$  by  $a$ .

For any point  $z$  of  $|\Delta(X)|$ , we will denote its 0-coloured representation by  $(x_z, u_z)$ . The point  $z$  belongs to  $|\Delta_E(X)|$  (resp.  $|\Delta_E(Y)|$ ) iff  $\Psi_X(x_z) \cap Irr(E) \neq \emptyset$  (resp. iff  $\Psi_X(x_z) \cap Irr(E) \neq \emptyset$  and  $\Psi_X(x_z) \subset Irr(Y)$ ).

For  $\emptyset \neq J \subset Irr(X)$ ,  $x$  a generic point of  $X_J$ , and  $u \in [0, 1]^J$ , we define the support  $Supp(u)$  of  $u$  as the set of indices  $i \in J$  with  $u(i) \neq 1$ . We identify  $(x, u)$  with the point  $z \in |\Delta(X)|$  with 0-coloured representation  $(x_z, u_z)$ , where  $u_z$  is the restriction of  $u$  to  $Supp(u)$  and  $x_z$  is the unique generic point of  $X_{Supp(u)}$  such that  $x$  belongs to the Zariski closure of  $\{x_z\}$  in  $X$ .

Consider the function

$$\Phi_Y^X : |\Delta_E(X)| \times [0, 1] \rightarrow |\Delta_E(X)| : (z = (x_z, u_z), \rho) \mapsto (x_z, u(\rho)) = \Phi_Y^X(z, \rho)$$

where  $u(\rho) \in [0, 1]^{\Psi_X(x_z)}$  depends on  $x_z$ ,  $u_z$  and  $\rho$ , and is defined in the following way.

*Case 1.* If  $a \notin \Psi_X(x_z)$ , then  $u(\rho) = u_z$  for each value of  $\rho$ .

*Case 2.* Now assume that  $a \in \Psi_X(x_z)$ . If there exists an element  $b \in \Psi_Y(x_z) = \Psi_X(x_z) \cap Irr(Y)$  with  $u_z(b) = 0$ , then we put

$$u(\rho)(i) = \begin{cases} u_z(i) & \text{for } i \in \Psi_Y(x_z) \text{ and } \rho \in [0, 1] \\ u_z(a) & \text{for } i = a \text{ and } \rho \in [0, 1/2] \\ u_z(a) + (1 - u_z(a))(2\rho - 1) & \text{for } i = a \text{ and } \rho \in [1/2, 1] \end{cases}$$

*Case 3.* Finally, suppose that  $u_z(i) \neq 0$  for all  $i \in \Psi_Y(x_z)$ . Then necessarily  $u_z(a) = 0$ . We put  $u(\rho)(a) = 0$  for  $\rho \in [0, 1/2]$ , and

$$u(\rho)(i) = u_z(i) - 2\rho \cdot \min_{j \in \Psi_X(x_z)} (u_z(j))$$

for  $i \in \Psi_Y(x_z)$  and  $\rho \in [0, 1/2]$ . Then there exists an element  $b \in \Psi_Y(x_z)$  with  $u(1/2)(b) = 0$ , so that the definition in Case 2 applies to  $v := u(1/2) \in [0, 1]^{\Psi_X(x_z)}$ , and we put  $u(\rho) = v(\rho)$  for  $\rho \in [1/2, 1]$ .

It is easily seen that the map  $\Phi_Y^X$  is continuous, and defines a strong deformation retract of  $|\Delta_E(X)|$  onto  $|\Delta_E(Y)|$ . Let us check the commutativity of diagram (3.1). Let  $z$  be any point of  $\mathscr{Y}$ , and put  $sp_{\mathscr{Y}}(z) = sp_X(z) = z_0$ . For each element  $C$  of  $\Psi_X(z_0)$ , we choose a generator  $T_C$  of the kernel of the natural morphism

$\mathcal{O}_{X,z_0} \rightarrow \mathcal{O}_{C,z_0}$ . Denote by  $y$  and  $x$  the points of  $\text{Str}(Y)$ , resp.  $\text{Str}(X)$  such that  $z_0$  belongs to the corresponding stratum of  $Y$ , resp.  $X$ . Then by Lemma 3.6 we have to show that  $\Phi_Y^X(\cdot, 1)$  takes the point  $z' = (\tau_X \circ g_\eta)(z)$  of  $|\Delta_E(X)|$  with 0-coloured representation  $(x, (|T_i(z)|)_{i \in \Psi_X(z_0)})$  to the point  $\tau_Y(z)$  of  $|\Delta_E(Y)|$  with 0-coloured representation  $(y, (|T_i(z)|)_{i \in \Psi_Y(z_0)})$ .

If  $\Psi_X(z_0) = \Psi_Y(z_0)$  this is obvious, so assume that  $\Psi_X(z_0) = \Psi_Y(z_0) \sqcup \{a\}$ . Since  $z$  is an element of  $\mathcal{Z}_\eta$ , there exists an element  $i$  of  $\Psi_Y(z_0)$  with  $|T_i(z)| = 0$ . By Case 2 of the definition,  $\Phi_Y^X(z', 1)$  is given by the couple  $(x, v)$  with  $v \in [0, 1]^{\Psi_X(z_0)}$ ,  $v(i) = |T_i(z)|$  for  $i \neq a$ , and  $v(a) = 1$ . By the identifications we made, this is exactly the point  $\tau_Y(z)$  (since  $y$  is the unique generic point of  $X_{\Psi_Y(z_0)}$  such that  $x$  belongs to the Zariski closure of  $\{y\}$  in  $Y$ ).  $\square$

**Corollary 3.8.** *The map  $h_\eta : E_\eta \rightarrow \mathfrak{E}_\eta$  is a homotopy equivalence.*

*Proof.* By Proposition 3.7,  $\Phi_E^X(\cdot, 1)$  is a homotopy equivalence with as homotopy inverse the natural embedding  $i : |\Delta(E)| \rightarrow |\Delta_E(X)|$ . Moreover,  $\tau_E$  and  $\tau_X$  are homotopy equivalences with as homotopy inverses the natural embeddings  $\sigma_E : |\Delta(E)| \rightarrow E_\eta$ , resp.  $\sigma_X : |\Delta_E(X)| \rightarrow \mathfrak{E}_\eta$  (we used Berkovich' natural homeomorphisms to identify  $|\Delta(E)|$  with the skeleton  $S(E)$  and  $|\Delta_E(X)|$  with  $S(X) \cap \text{sp}_X^{-1}(E)$ ). Now the fact that  $h_\eta$  is a homotopy equivalence follows from the commutativity of the diagram in the statement of Proposition 3.7, and the commutativity of the diagram

$$\begin{array}{ccc} |\Delta(E)| & \xrightarrow{i} & |\Delta_E(X)| \\ \sigma_E \downarrow & & \downarrow \sigma_X \\ E_\eta & \xrightarrow{h_\eta} & \mathfrak{E}_\eta \end{array}$$

which follows easily from the description of the skeleton in Section 3.3.  $\square$

#### 4. HOMOTOPY TYPE OF THE ANALYTIC MILNOR FIBER

Let  $k$  be any field, and put  $R = k[[t]]$  and  $K = k((t))$ , endowed with the  $t$ -adic valuation (so  $R = K^o$ ). For each  $0 < r < 1$ , we denote by  $|\cdot|_r$  the  $t$ -adic absolute value on  $K$  with  $|t|_r = r$ . We fix an algebraic closure  $K^a$  of  $K$ .

We endow  $k$  with its trivial absolute value  $|\cdot|_0$ , and we put  $\mathfrak{R} := \text{Spf } R$ . Moreover, we endow  $k[t]$  with the trivial Banach norm (this norm coincides with the Gauss norm if we view  $k[t]$  as the algebra of convergent power series  $k\{t\}$ ). The formal scheme  $\mathfrak{R}$  is a special formal  $k$ -scheme in the sense of [6]. We denote its generic fiber by  $\mathfrak{R}_\eta$ ; it coincides with the open unit disc  $D_0(1) = \{x \in \mathcal{M}(k[t]) \mid |t(x)| < 1\}$ .

The following result was stated in Step 3 of the proof of [3, Thm 4.1], without proof. We include the elementary proof for the reader's convenience.

**Lemma 4.1.** *The natural map*

$$\Psi : D_0(1) \rightarrow [0, 1[ : x \mapsto |t(x)|$$

*is a homeomorphism. If we put  $p_r := \Psi^{-1}(r)$  for  $0 \leq r < 1$ , then the residue field  $\mathcal{H}(p_r)$  is  $K_0 = (k, |\cdot|_0)$  for  $r = 0$ , and  $K_r = (K, |\cdot|_r)$  for  $0 < r < 1$ .*

*Proof.* The map  $\Psi$  is obviously continuous. Its inverse is

$$\Psi^{-1} : [0, 1[ \rightarrow D_0(1) : r \mapsto p_r$$

where  $p_r$  is bounded multiplicative semi-norm in  $\mathcal{M}(k[t])$  defined by  $p_r(\sum_{i=0}^n a_i t^i) := \max_i |a_i|_0 r^i$  (with the convention that  $0^0 = 1$ ).

Indeed: it is clear that  $\Psi \circ \Psi^{-1}$  is the identity. Now we show that  $\Psi^{-1}$  is also left inverse to  $\Psi$ . Choose  $x$  in  $D_0(1)$  and put  $r = |t(x)|$ . A classical trick shows that the residue field  $\mathcal{H}(x)$  of  $x$  is ultrametric: for  $f, g$  in  $k[t]$ , and  $n \geq 0$ , we have

$$\begin{aligned} |(f+g)^n(x)| &= |(\sum_{i=0}^n \binom{n}{i} f^i g^{n-i})(x)| \leq \sum_{i=0}^n |\binom{n}{i}|_0 |f(x)|^i |g(x)|^{n-i} \\ &\leq (n+1)(\max\{|f(x)|, |g(x)|\})^n \end{aligned}$$

and taking  $n$ -th roots and sending  $n$  to  $\infty$ , we see that  $|(f+g)(x)| \leq \max\{|f(x)|, |g(x)|\}$ .

If  $r = 0$ , then the ultrametric property implies  $x = p_0$ , so we may assume  $0 < r < 1$ . In this case,  $|at^i(x)| \neq |a't^j(x)|$  for  $a, a' \in k^*$  and  $i \neq j$ , so we get  $|(\sum_{i=0}^n a_i t^i)(x)| = \max_i |a_i|_0 r^i$ .

It remains to prove that  $\Psi^{-1}$  is continuous. By definition of the spectral topology, it suffices to show that  $[0, 1[ \rightarrow \mathbb{R}_+ : r \mapsto |f(p_r)| = p_r(f)$  is continuous for each  $f \in k[t]$ . This, however, is clear.

Finally, we determine the residue fields of the points  $p_r$ . Since  $|\sum_{i=0}^n a_i t^i(x)| = |a_0|_0$ , we see that  $\mathcal{H}(p_0) = (k, |\cdot|_0)$ . For  $r > 1$ , no element of  $k[t] \setminus \{0\}$  vanishes in  $p_r$ , so  $\mathcal{H}(p_r)$  is the completion of  $k(t)$  w.r.t.  $p_r$ ; this is exactly  $(K, |\cdot|_r)$ .  $\square$

If  $\mathcal{X}$  is a special formal  $R$ -scheme, then we can also consider  $\mathcal{X}$  as a special formal  $k$ -scheme via the composition  $\mathcal{X} \rightarrow \mathfrak{R} \rightarrow \text{Spec } k$ . We denote this object by  $\mathcal{X}^k$ . This yields a forgetful functor

$$(SpF/R) \rightarrow (SpF/k) : \mathcal{X} \mapsto \mathcal{X}^k$$

from the category  $(SpF/R)$  of special formal  $R$ -schemes to the category  $(SpF/k)$  of special formal  $k$ -schemes. We can associate to  $\mathcal{X}^k$  its generic fiber  $\mathcal{X}_\eta^k$  (a  $K_0$ -analytic space), and there is a natural specialization map  $sp_{\mathcal{X}^k} : \mathcal{X}_\eta^k \rightarrow \mathcal{X}_0$ .

The map of special formal  $k$ -schemes  $h : \mathcal{X}^k \rightarrow \mathfrak{R}$  induces a map of  $K_0$ -analytic spaces  $h_\eta : \mathcal{X}_\eta^k \rightarrow \mathfrak{R}_\eta$ . Its fibers can be described as follows.

**Lemma 4.2.** *For  $0 \leq r < 1$ , there is a canonical isomorphism*

$$\mathcal{X}(r) \cong \mathcal{X}^k \widehat{\otimes}_{\mathfrak{R}} \text{Spf } \mathcal{H}(p_r)^o$$

*and the fiber of  $h_\eta$  over  $p_r$  is canonically isomorphic to the  $K_r$ -analytic space  $\mathcal{X}(r)_\eta$ .*

*Proof.* By the construction of the generic fiber in [6, §1], it suffices to consider the case where  $\mathcal{X} = \text{Spf } A$ , with  $A$  of the form  $R[[x_1, \dots, x_m]][y_1, \dots, y_n]$ . Then  $\mathcal{X}_\eta^k$  is the polydisc

$$\{z \in \mathcal{M}(k[t, x_1, \dots, y_n]) \mid |t(z)| < 1 \text{ and } |x_i(z)| < 1 \text{ for } i = 1, \dots, m\}$$

where  $k[t, x_1, \dots, y_n]$  carries the trivial Banach norm, and the map  $h_\eta$  sends the bounded multiplicative semi-norm  $z$  to its restriction to  $k[t]$ . Now we observe that

$$k[t, x_1, \dots, y_n] \widehat{\otimes}_{k[t]} \mathcal{H}(p_0) \cong k[x_1, \dots, y_n]$$

while

$$k[t, x_1, \dots, y_n] \widehat{\otimes}_{k[t]} \mathcal{H}(p_r) \cong K_r\{x_1, \dots, y_n\}$$

for  $0 < r < 1$ .  $\square$

**Proposition 4.3.** *Let  $\mathcal{X}$  be a special formal  $R$ -scheme. If we denote by  $\lambda$  the map  $|\mathcal{X}_\eta^k| \rightarrow |\mathfrak{R}_\eta| \cong [0, 1[$ , then for each  $r \in [0, 1[$ , there exists a canonical homeomorphism  $|\mathcal{X}(r)_\eta| \cong \lambda^{-1}(r)$  such that the square*

$$\begin{array}{ccc} |\mathcal{X}(r)_\eta| & \xrightarrow{\sim} & \lambda^{-1}(r) \subset |\mathcal{X}_\eta^k| \\ \text{\scriptsize $sp_{\mathcal{X}(r)}$} \downarrow & & \downarrow \text{\scriptsize $sp_{\mathcal{X}^k}$} \\ |\mathcal{X}_0| & \xlongequal{\quad} & |\mathcal{X}_0| \end{array}$$

*commutes.*

*Moreover, for any  $0 < r < 1$ , there exists a canonical homeomorphism*

$$\phi : \lambda^{-1}(]0, 1[) \rightarrow |\mathcal{X}(r)_\eta| \times ]0, 1[$$

*such that the composition of  $\phi$  with the projection  $|\mathcal{X}(r)_\eta| \times ]0, 1[ \rightarrow ]0, 1[$  coincides with  $\lambda : \lambda^{-1}(]0, 1[) \rightarrow ]0, 1[$ .*

*Proof.* It suffices to consider the case where  $\mathcal{X} = \text{Spf } A$  is affine, with  $A$  of the form

$$A = R[[x_1, \dots, x_m]][y_1, \dots, y_n]/(f_1, \dots, f_\ell)$$

Then  $\mathcal{X}_\eta^k$  is a closed subset of the polydisc

$$E := \{z \in \mathcal{M}(k[t, x_1, \dots, y_n]) \mid |t(z)| < 1 \text{ and } |x_i(z)| < 1 \text{ for } i = 1, \dots, m\}$$

where  $k[t, x_1, \dots, y_n]$  carries the trivial Banach norm, and this closed subset is defined by the equations  $z(f_j) = 0$ ,  $j = 1, \dots, \ell$ . The map  $\lambda$  sends  $z$  to the point  $|t(z)|$  of  $|\mathfrak{R}_\eta| \cong [0, 1[$ .

It is clear that  $\lambda^{-1}(r)$  is canonically homeomorphic to  $|\mathcal{X}(r)_\eta|$  for each  $r$ , and that this homeomorphism is compatible with the specialization maps  $sp_{\mathcal{X}(r)}$  and  $sp_{\mathcal{X}^k}$ : w.r.t. both of these maps, the image of a point  $z$  in  $\lambda^{-1}(r)$  is the open prime ideal  $\{f \in A \mid |f(z)| < 1\}$  of  $A$ .

Now fix  $r$  in  $]0, 1[$ , and consider the map

$$\Psi : \lambda^{-1}(]0, 1[) \rightarrow |\mathcal{X}(r)_\eta| \times ]0, 1[ : x \mapsto (x^{\log_{\lambda(x)} r}, \lambda(x))$$

where we denote by  $x^{\log_{\lambda(x)} r}$  the bounded multiplicative semi-norm in  $\mathcal{M}(k[t, x_1, \dots, y_n])$  sending  $f \in k[t, x_1, \dots, y_n]$  to  $x(f)^{\log_{\lambda(x)} r}$ ; then clearly  $\lambda(x^{\log_{\lambda(x)} r}) = r$ . The map  $\Psi$  is a bijection, with inverse

$$\Psi^{-1} : |\mathcal{X}(r)_\eta| \times ]0, 1[ \rightarrow \lambda^{-1}(]0, 1[) : (x, \rho) \mapsto x^{\log_r \rho}$$

and one checks immediately that both  $\Psi$  and  $\Psi^{-1}$  are continuous.  $\square$

**Proposition 4.4.** *Let  $\mathcal{X}$  be a strictly semi-stable formal  $R$ -scheme, let  $E$  be a union of irreducible components of  $\mathcal{X}_s$ , and denote by  $\mathfrak{E}$  the formal completion of  $\mathcal{X}$  along  $E$ .*

*(1) The inclusion map*

$$\mathfrak{E}(r)_\eta \rightarrow \mathfrak{E}_\eta^k$$

*is a homotopy equivalence, for each  $r \in [0, 1[$ .*

*(2) The natural map  $E_\eta \rightarrow \mathfrak{E}(0)_\eta$ , induced by the closed immersion  $E \rightarrow \mathfrak{E}(0)$ , is a homotopy equivalence. In particular, if  $E$  is proper, then  $E^{\text{an}} \rightarrow \mathfrak{E}(0)_\eta$  is a homotopy equivalence.*

*Proof.* (1) In the last part of the proof of [3, 4.1], Berkovich constructs the so-called skeleton  $S(\mathcal{X}/\mathfrak{R}) \subset \mathcal{X}_\eta^k$  associated to the morphism of special formal  $k$ -schemes  $\mathcal{X}^k \rightarrow \mathfrak{R}$ : it is the union of the skeletons  $S(\mathcal{X}(r)) \subset \mathcal{X}(r)_\eta \cong \lambda^{-1}(r)$ ,  $r \in [0, 1[$ . Moreover, he shows that the map  $\Phi : \mathcal{X}_\eta^k \times [0, 1] \rightarrow \mathcal{X}_\eta^k$  which coincides with  $\Phi_{\mathcal{X}(r)}$  on  $\lambda^{-1}(r) \times [0, 1]$ , is a strong deformation retract of  $\mathcal{X}_\eta^k$  onto  $S(\mathcal{X}/\mathfrak{R})$ . By [7, 5.2(iv)]  $\Phi$  restricts to a strong deformation retract of  $(sp_{\mathcal{X}^k})^{-1}(E)$  onto  $S(\mathcal{X}/\mathfrak{R}) \cap (sp_{\mathcal{X}^k})^{-1}(E)$  compatible with  $\lambda$  (i.e. it is a strong deformation retract on each fiber of  $\lambda$ ).

Next, Berkovich states that there exists a homeomorphism  $|\Delta(\mathcal{X}_s)| \times \mathfrak{R}_\eta \rightarrow S(\mathcal{X}/\mathfrak{R})$  such that the projection of  $|\Delta(\mathcal{X}_s)| \times \mathfrak{R}_\eta$  on the second factor  $\mathfrak{R}_\eta$  corresponds to the map  $\lambda$  on  $S(\mathcal{X}/\mathfrak{R})$ . His construction contains a minor error, but it can easily be corrected as follows. By the same arguments, it suffices to construct a homeomorphism

$$f : \Sigma_0^n \times [0, 1[ \rightarrow \{(x, \rho) \in \mathbb{R}^{[n]} \times [0, 1[ \mid x \in \Sigma_\rho^n\}$$

for each  $n$ , such that these homeomorphisms are compatible with the face maps (so that we get good gluing properties). We can define  $f$  by sending  $(y, \rho)$  to  $(x, \rho)$ , where  $x$  is the unique intersection point of  $\Sigma_\rho^n$  and the segment in  $\mathbb{R}^{[n]}$  joining  $y$  and  $(1, \dots, 1)$ . This homeomorphism identifies  $S(\mathcal{X}/\mathfrak{R}) \cap (sp^k)^{-1}(E)$  with  $|\Delta_E(X)| \times \mathfrak{R}_\eta$ . This proves (1).

(2) Since  $\mathcal{X}_s$  is a strictly semi-stable  $k$ -variety, and  $\mathfrak{E}(0)$  is isomorphic to the completion of  $\mathcal{X}_s$  along  $E$ , this follows immediately from Corollary 3.8 and the fact that  $E_\eta \cong E^{an}$  for proper  $E$ .  $\square$

**Lemma 4.5.** *Assume that  $k$  is algebraically closed. If  $\mathcal{X}$  is a strictly semi-stable formal  $R$ -scheme, and  $E$  is any strata subset of  $\mathcal{X}_s$ , then the natural map*

$$sp_{\mathcal{X}(r)}^{-1}(E) \widehat{\times}_{K_r} L \rightarrow sp_{\mathcal{X}(r)}^{-1}(E)$$

*is a homotopy equivalence for any  $r \in ]0, 1[$  and any isometric embedding of non-archimedean fields  $K_r \subset L$ .*

*Proof.* Put  $\mathcal{Y} = \mathcal{X} \widehat{\times}_{\text{Spf } R} \text{Spf } L^\circ$ . Then  $\mathcal{Y}_\eta \cong \mathcal{X}(r)_\eta \widehat{\times}_{K_r} L$ ,  $\mathcal{Y}_s \cong \mathcal{X}_s \times_k \widetilde{L}$ , and these natural isomorphisms commute with the specialization maps  $sp_{\mathcal{X}(r)}$  and  $sp_{\mathcal{Y}}$ , so that they induce a natural isomorphism

$$sp_{\mathcal{X}(r)}^{-1}(E) \widehat{\times}_{K_r} L \cong sp_{\mathcal{Y}}^{-1}(F)$$

where  $F$  denotes the inverse image of  $E$  in  $\mathcal{Y}_s$ .

Since  $k$  is algebraically closed, the natural map  $h_s : \mathcal{Y}_s \rightarrow \mathcal{X}_s$  induces a bijection  $\text{Irr}(\mathcal{Y}_s) \cong \text{Irr}(\mathcal{X}_s)$  and a homeomorphism  $\alpha : |\Delta(\mathcal{Y}_s)| \cong |\Delta(\mathcal{X}_s)|$  identifying  $|\Delta_F(\mathcal{Y}_s)|$  with  $|\Delta_E(\mathcal{X}_s)|$ . If we denote by  $h$  the natural morphism  $\mathcal{Y}_\eta \rightarrow \mathcal{X}(r)_\eta$ , it is easy to see from the description in Section 3.3 that the diagram

$$\begin{array}{ccccccc} sp_{\mathcal{Y}}^{-1}(F) & \xrightarrow{\tau_{\mathcal{Y}}} & |\Delta(\mathcal{Y}_s)| & \xrightarrow{\cong} & S(\mathcal{Y}) \cap sp_{\mathcal{Y}}^{-1}(F) & \longrightarrow & sp_{\mathcal{Y}}^{-1}(F) \\ h \downarrow & & \alpha \downarrow \cong & & \cong \downarrow h & & \downarrow h \\ sp_{\mathcal{X}(r)}^{-1}(E) & \xrightarrow{\tau_{\mathcal{X}(r)}} & |\Delta(\mathcal{X}_s)| & \xrightarrow{\cong} & S(\mathcal{X}(r)) \cap sp_{\mathcal{X}(r)}^{-1}(E) & \longrightarrow & sp_{\mathcal{X}(r)}^{-1}(E) \end{array}$$

commutes (the right horizontal arrows are the inclusion maps).  $\square$

**Proposition 4.6.** *Assume that  $k$  is an algebraically closed field of characteristic zero, and fix  $r \in ]0, 1[$ . Let  $X$  be a proper flat  $R$ -variety such that  $X \times_R K$  is smooth over  $K$ , and such that  $X_s$  has at most one singular point  $x$ , and denote by  $\mathcal{X}$  its  $t$ -adic completion. Then there exists a canonical long exact sequence in integral singular cohomology*

$$\dots \rightarrow H^i((X_s)^{an}, \mathbb{Z}) \rightarrow H^i(\overline{\mathcal{X}(r)}_\eta, \mathbb{Z}) \xrightarrow{i^*} \tilde{H}^i(\overline{]x[}, \mathbb{Z}) \rightarrow H^{i+1}((X_s)^{an}, \mathbb{Z}) \rightarrow \dots$$

with  $]x[ = sp_{\mathcal{X}(r)}^{-1}(x)$ , and where  $i : \overline{]x[} \rightarrow \overline{\mathcal{X}(r)}_\eta$  is the inclusion map and  $\tilde{H}^*(\cdot)$  is reduced cohomology.

*Proof.* Passing to a finite extension of  $R$ , we may assume that there exists a proper morphism of  $R$ -varieties  $h : Y \rightarrow X$  such that  $Y$  is strictly semi-stable, such that  $h$  is an isomorphism over the complement of  $x$  in  $X$ , and such that  $E := h^{-1}(x)$  is a union of irreducible components of  $Y_s$ . We denote by  $\mathcal{Y}$  the  $t$ -adic completion of  $Y$ , and by  $\mathfrak{E}$  the formal completion of  $Y$  along  $E$ .

The morphism  $h$  induces a surjective morphism of  $k$ -analytic spaces  $h_s^{an} : (Y_s)^{an} \rightarrow (X_s)^{an}$ ; since  $X$  and  $Y$  are proper over  $R$ ,  $(Y_s)^{an}$  and  $(X_s)^{an}$  are compact Hausdorff spaces. Moreover,  $h_s^{an}$  maps  $(Y_s \setminus E)^{an} \cong (Y_s)^{an} \setminus E^{an}$  isomorphically to  $(X_s)^{an} \setminus \{x\}$ , and maps  $E^{an}$  to  $\{x\}$ . Therefore,  $h_s^{an}$  induces a homeomorphism  $(Y_s)^{an}/E^{an} \approx (X_s)^{an}$ , and we get a natural exact sequence

$$\dots \rightarrow H^i((X_s)^{an}, \mathbb{Z}) \rightarrow H^i((Y_s)^{an}, \mathbb{Z}) \rightarrow \tilde{H}^i(E^{an}, \mathbb{Z}) \rightarrow H^{i+1}((X_s)^{an}, \mathbb{Z}) \rightarrow \dots$$

By Proposition 4.4 and Lemma 4.5, the natural maps  $\overline{\mathcal{X}(r)}_\eta \rightarrow \mathcal{Y}_\eta^k$ ,  $(Y_s)^{an} \rightarrow \mathcal{Y}_\eta^k$ ,  $E^{an} \rightarrow \mathfrak{E}_\eta^k$  and  $\overline{]x[} \cong \overline{\mathfrak{E}(r)}_\eta \rightarrow \mathfrak{E}_\eta^k$  are all homotopy equivalences. Hence, we obtain the desired exact sequence.  $\square$

Now we come to the main result of this section: the description of the homotopy type of the analytic Milnor fiber. First, we need an auxiliary definition. Let  $r \in ]0, 1[$  be the value fixed in Section 2.3 to define the analytic Milnor fiber  $\mathcal{F}_x$  as a  $K_r$ -analytic space.

**Definition 4.7** (Strictly semi-stable model). *Let  $X$  be a variety over  $k$ , endowed with a morphism  $f : X \rightarrow \text{Spec } k[t]$  which is flat over the origin and has smooth generic fiber, and let  $x$  be a closed point of the special fiber  $X_s = f^{-1}(0)$ . A strictly semi-stable model of the germ  $(f, x)$  of  $f$  at  $x$  consists of the following data:*

- (1) an integer  $d > 0$ , and an embedding of  $k[t]$ -algebras

$$A_d = k[t, u]/(u^d - t) \rightarrow K^a$$

- (2) a flat projective morphism  $g : Y \rightarrow \text{Spec } A_d$  whose  $u$ -adic completion is strictly semi-stable,
- (3) open subschemes  $U$  and  $V$  of  $X$ , resp.  $Y$ , with  $x \in U$ ,
- (4) a proper morphism  $\varphi : V \rightarrow U \times_{k[t]} A_d$  which is an isomorphism over the complement of  $U_s$ , such that  $g = p_2 \circ \varphi$  on  $V$  (with  $p_2$  the projection  $U \times_{k[t]} A_d \rightarrow \text{Spec } A_d$ ) and such that  $\varphi^{-1}(x)$  is a union of irreducible components of the special fiber  $Y_s$  of  $g$ .

We'll denote this strictly semi-stable model by  $(Y, g, \varphi)$  (the other data are implicit in the notation). We call  $d$  the ramification index of the strictly semi-stable model.

A strictly semi-stable model of the analytic Milnor fiber  $\mathcal{F}_x$  of  $f$  at  $x$  consists of the following data:



- (1) an integer  $d > 0$ , and an embedding of non-archimedean  $K_r$ -fields

$$K_r(d) = K_r[u]/(u^d - t) \rightarrow K_r^a$$

- (2) a strictly semi-stable formal  $K_r(d)^\circ$ -scheme  $\mathcal{Y}$ , and a closed subvariety  $E$  of  $\mathcal{Y}_s$  which is a union of irreducible components of  $\mathcal{Y}_s$ ,  
 (3) an isomorphism of  $K_r(d)$ -analytic spaces  $\varphi : sp_{\mathcal{Y}(r)}^{-1}(E) \cong \mathcal{F}_x \times_{K_r} K_r(d)$ .

We'll denote this strictly semi-stable model by  $(\mathcal{Y}, E, \varphi)$  (the other data are implicit in the notation). We call  $d$  the ramification index of the strictly semi-stable model.

It is clear that any strictly semi-stable model of  $(f, x)$  induces a strictly semi-stable model of  $\mathcal{F}_x$  by passing to the  $t$ -adic completion (and of course, this still holds if we omit the projectivity condition in (2)).

**Proposition 4.8.** *Suppose that  $k$  has characteristic zero. Let  $X$  be a variety over  $k$ , endowed with a morphism  $f : X \rightarrow \text{Spec } k[t]$  which is flat over the origin and has smooth generic fiber, and let  $x$  be a closed point of the special fiber  $X_s = f^{-1}(0)$ . Then  $(f, x)$  admits a strictly semi-stable model.*

*Proof.* We may as well assume that  $f$  is projective: restrict  $f$  to an affine neighbourhood of  $x$ , consider its projective completion, and resolve singularities at infinity. Now the result follows from the semi-stable reduction theorem [16, II].  $\square$

**Corollary 4.9.** *Under the same conditions,  $\mathcal{F}_x$  admits a strictly semi-stable model.*

**Theorem 4.10.** *Suppose that  $k$  is algebraically closed (of arbitrary characteristic). Let  $X$  be a variety over  $k$ , endowed with a morphism  $f : X \rightarrow \text{Spec } k[t]$  which is flat over the origin and has smooth generic fiber, and let  $x$  be a closed point of the special fiber  $X_s = f^{-1}(0)$ . Suppose that  $\mathcal{F}_x$  admits a strictly semi-stable model  $(\mathcal{Y}, E, \varphi)$ . Then  $\overline{\mathcal{F}_x}$  is naturally homotopy-equivalent to  $|\Delta(E)|$ . In particular, the homotopy type of  $|\Delta(E)|$  does not depend on the chosen strictly semi-stable model.*

*Proof.* Let  $d$  be the ramification index of the strictly semi-stable model  $(\mathcal{Y}, E, \varphi)$ . The isomorphism  $\varphi$  induces an isomorphism

$$sp_{\mathcal{Y}(r)}^{-1}(E) \widehat{\times}_{K_r(d)} \widehat{(K_r)^a} \cong \overline{\mathcal{F}_x}$$

so the result follows from Lemma 4.5 and Proposition 4.4.  $\square$

**Remark.** By the same arguments, we have the following result: suppose that  $k$  is algebraically closed, and fix  $r \in ]0, 1[$ . Consider a generically smooth *stft* formal  $R$ -scheme  $\mathcal{X}$ , and assume that it admits a strictly semi-stable model  $h : \mathcal{Y} \rightarrow \mathcal{X}$  (i.e.  $\mathcal{Y}$  is a strictly semi-stable formal  $L^\circ$ -scheme for some finite extension  $L$  of  $K_r$  in  $K_r^a$ , and  $h$  is a morphism of formal  $R$ -schemes such that the induced morphism  $\mathcal{Y}_\eta \rightarrow \mathcal{X}(r)_\eta \times_{K_r} L$  is an isomorphism). Such a model exists, in particular, if  $k$  has characteristic zero (use embedded resolution for singularities for  $(\mathcal{X}, X_s)$  as in [26] and apply the algorithm for semi-stable reduction in characteristic zero [16, II]).

The analytic space  $\overline{\mathcal{X}(r)}_\eta$  is naturally homotopy-equivalent to  $|\Delta(\mathcal{Y}_s)|$ . In particular, the homotopy type of  $|\Delta(\mathcal{Y}_s)|$  does not depend on the chosen strictly semi-stable model. This result, and the one in Theorem 4.10, are similar in nature to [27, Thm 4.8].  $\square$

## 5. WEIGHT ZERO PART OF THE MIXED HODGE STRUCTURE ON THE NEARBY COHOMOLOGY

**5.1. Cocubical systems.** We recall the following definition. For any finite, non-empty set  $S$ , we denote by  $\square_S$  the set of *non-empty* subsets of  $S$ , ordered by inclusion. For any category  $\mathcal{C}$ , the category of  $S$ -cocubical systems in  $\mathcal{C}$  is the category of covariant functors  $\square_S \rightarrow \mathcal{C}$  (with natural transformations as morphisms).

Let  $\mathcal{A}$  be an abelian category, and denote by  $C^+(\mathcal{A})$  the category of bounded below complexes in  $\mathcal{A}$ . We fix an integer  $n \geq 0$ , and we consider a  $[n]$ -cocubical system  $(C_L^\bullet)_{L \in \square_{[n]}}$  in  $C^+(\mathcal{A})$ . For each object  $L$  in  $\square_{[n]}$  and each  $p \in \mathbb{Z}$ , we denote by  $d_L : C_L^p \rightarrow C_L^{p+1}$  the differential in the complex  $C_L^\bullet \in C^+(\mathcal{A})$ . For each couple  $(L, L')$  of objects in  $\square_{[n]}$  with  $L \subset L'$ , we denote by

$$\delta_{L'}^L : C_L^\bullet \rightarrow C_{L'}^\bullet$$

the face map which is part of the cocubical system. We define the associated simple complex  $s_\square((C_L^\bullet)_{L \in \square_{[n]}})$  as follows (we use the notation  $s_\square$  to avoid confusion with the simple complex  $s(\cdot)$  associated to a double complex).

First, we define a double complex  $A^{\bullet\bullet}$ . We put

$$A^{p,q} = \begin{cases} 0 & \text{for } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{<0} \\ \oplus_{|L|=q+1} C_L^p & \text{for } (p,q) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \end{cases}$$

The horizontal differential  $A^{p,q} \rightarrow A^{p+1,q}$ , for  $q \geq 0$ , is given by

$$\oplus_{|L|=q+1} \{d_L : C_L^p \rightarrow C_L^{p+1}\}$$

The restriction of the vertical differential  $A^{p,q} \rightarrow A^{p,q+1}$  to the component  $C_L^p$  is given by

$$C_L^p \rightarrow A^{p,q+1} : \sum_{i \in [n] \setminus L} (-1)^{\varepsilon(L,i)+p+1} \delta_{L \cup \{i\}}^L$$

where  $\varepsilon(L,i)$  denotes the number of elements  $j$  in  $L$  with  $j < i$ .

We define  $s_\square((C_L^\bullet)_{L \in \square_{[n]}})$  as the associated simple complex  $s(A^{\bullet\bullet})$ ; we'll also denote it by  $s_\square(C_L^\bullet)$  to simplify notation. A map  $f_L : C_L^\bullet \rightarrow D_L^\bullet$  of  $[n]$ -cocubical systems in  $C^+(\mathcal{A})$  induces a map  $s_\square(f_L) : s_\square(C_L^\bullet) \rightarrow s_\square(D_L^\bullet)$  between the associated simple complexes, so we get a functor  $s_\square(\cdot)$  from the category of  $[n]$ -cocubical systems in  $C^+(\mathcal{A})$  to the category  $C^+(\mathcal{A})$ .

Denote by  $C^+(\mathcal{A}, W, F)$  the category of bifiltered bounded below complexes in  $\mathcal{A}$  (with  $F$  decreasing and  $W$  increasing). If  $(C_L^\bullet, W, F)_{L \in \square_{[n]}}$  is a  $[n]$ -cocubical system in  $C^+(\mathcal{A}, W, F)$  then we endow the simple complex  $C^\bullet = s_\square(C_L^\bullet)$  with the following filtrations: for each  $n, r \in \mathbb{Z}$ , we put

$$\begin{aligned} F^r C^n &= \oplus_{p+q=n, q \geq 0} \oplus_{|L|=q+1} F^r C_L^p \\ W_r C^n &= \oplus_{p+q=n, q \geq 0} \oplus_{|L|=q+1} W_{r+q} C_L^p \end{aligned}$$

We call the bifiltered complex  $(s_\square(C_L^\bullet), W, F)$  the associated simple complex of the cocubical system  $(C_L^\bullet, W, F)_{L \in \square_{[n]}}$ , and we put

$$s_\square(C_L^\bullet, W, F) = (s_\square(C_L^\bullet), W, F)$$

This defines a functor  $s_\square$  from the category of  $[n]$ -cocubical systems in  $C^+(\mathcal{A}, W, F)$  to the category  $C^+(\mathcal{A}, W, F)$ .

Let us consider some elementary examples, which will be of use later on.

**Example 5.1.** Let  $X$  be a smooth complex variety and let  $E$  be a proper strict normal crossing divisor on  $X$ , with irreducible components  $E_i$ ,  $i = 0, \dots, n$ . For any  $L \in \square_{[n]}$ , we put  $E_L = \cap_{i \in L} E_i$  and denote by  $a_L : E_L \rightarrow X$  the inclusion. Consider the bifiltered complex  $\mathcal{H}_L = ((a_L)_* \Omega_{E_L}^\bullet, W, F)$  where  $F$  is the stupid filtration, and  $W$  is given by

$$W_i((a_L)_* \Omega_{E_L}^\bullet) = \begin{cases} 0 & \text{for } i < 0 \\ (a_L)_* \Omega_{E_L}^\bullet & \text{for } i \geq 0 \end{cases}$$

In other words,  $\mathcal{H}_L$  is the image under  $(a_L)_*$  of the complex component  $\text{Hdg}^\bullet(E_L)_\mathbb{C}$  of the mixed Hodge complex of sheaves associated to the smooth and proper complex variety  $E_L$ . If  $L'$  is another subset of  $[n]$  and  $L \subset L'$ , then the closed immersion  $E_{L'} \rightarrow E_L$  induces restriction maps  $\mathcal{H}_L \rightarrow \mathcal{H}_{L'}$ , which make  $(\mathcal{H}_L)_{L \in \square_{[n]}}$  into a  $[n]$ -cocubical system. We denote its associated simple complex by  $\text{Hdg}^\bullet(E)_\mathbb{C}$ . It is the complex component of a mixed Hodge complex of sheaves which induces the canonical mixed Hodge structure on  $H^*(E, \mathbb{Z})$  [24, 3.5].

**Example 5.2.** Let  $X$  be any topological space, and consider a cover  $\{E_i \mid i \in [n]\}$  of  $X$  by closed subsets. For any  $L \in \square_{[n]}$ , we put  $E_L = \cap_{i \in L} E_i$  and denote by  $a_L : E_L \rightarrow X$  the inclusion.

Let  $G^\bullet$  be any object in  $C^+(X)$  (i.e. a bounded below complex of abelian sheaves on  $X$ ), and consider the  $[n]$ -cocubical system defined by  $G_L^\bullet = (a_L)_*(a_L)^*G^\bullet$ . By adjunction, we have a natural map of complexes

$$G^\bullet \rightarrow \oplus_{i \in [n]} (a_i)_*(a_i)^*G^\bullet$$

where we wrote  $a_i$  instead of  $a_{\{i\}}$ . The target of this map is a direct summand of  $s_\square(G_L^\bullet)$ , so we get a map of complexes  $G^\bullet \rightarrow s_\square(G_L^\bullet)$ . This is a quasi-isomorphism, since for each  $q \geq 0$ , the  $q$ -th row of the double complex  $A^{\bullet\bullet}$  associated to the cocubical system  $G_L^\bullet$  is a resolution of  $G^q$ . If  $f : G^\bullet \rightarrow H^\bullet$  is a morphism in  $C^+(X)$ , then  $f$  induces a morphism of cocubical systems  $f_L : G_L^\bullet \rightarrow H_L^\bullet$ , and the square

$$\begin{array}{ccc} G^\bullet & \xrightarrow{f} & H^\bullet \\ \downarrow & & \downarrow \\ s_\square(G_L^\bullet) & \xrightarrow{s_\square(f_L)} & s_\square(H_L^\bullet) \end{array}$$

commutes.

**Example 5.3.** We keep the notations of Example 5.2, supposing moreover that  $X$  is a smooth complex variety, and that  $E_L$  is a smooth closed subvariety for each  $L \in \square_{[n]}$ . We consider the  $[n]$ -cocubical system of complexes

$$((a_L)_* \Omega_{E_L}^\bullet)_{L \in \square_{[n]}}$$

in  $C^+(X, \mathbb{C})$ . The product of restriction maps  $\Omega_X^\bullet \rightarrow \oplus_{i \in [n]} (a_i)_* \Omega_{E_i}^\bullet$  induces a map of complexes  $\Omega_X^\bullet \rightarrow s_\square((a_L)_* \Omega_{E_L}^\bullet)$ . This map is a quasi-isomorphism: via the quasi-isomorphisms  $\Omega_X^\bullet \cong \mathbb{C}_X$  and  $\Omega_X^\bullet \cong \mathbb{C}_{E_L}$  and the exactness of the functor  $(a_L)_* : C^+(E_L, \mathbb{C}) \rightarrow C^+(X, \mathbb{C})$ , we recover the situation of Example 5.2, with  $G^\bullet = \mathbb{C}_X$ .

**5.2. Localized limit mixed Hodge complex.** Let  $S$  be the open complex unit disc, and let  $f : X \rightarrow S$  be a projective morphism, with  $X$  a complex manifold. We fix a complex coordinate  $t$  on  $S$ . Assume that the special fiber  $X_s$  is a reduced strict normal crossing divisor  $E = \sum_{i \in I} E_i$ . We fix a total order on  $I$ , i.e. a bijection  $I \cong [a]$  for some integer  $a \geq 0$ . Let  $J$  be a non-empty subset of  $I$ , put  $E(J) = \cup_{i \in J} E_i$ , and denote by  $v_J : E(J) \rightarrow X_s$  the inclusion map. The total order on  $I$  induces a total order on  $J$ , i.e. a bijection  $J \cong [b]$  for some integer  $b \geq 0$ . These total orders are necessary to apply the functor  $s_\square$  to  $I$ - and  $J$ -cocubical systems.

In [17] (see also [18]), Navarro Aznar constructed a localized limit integral mixed Hodge complex of sheaves on  $E(J)$ , whose integral component is quasi-isomorphic to  $v_J^* R\psi_f(\mathbb{Z})$  (here  $R\psi_f$  is the complex analytic nearby cycle functor associated to  $f$ ). This mixed Hodge complex induces a canonical integral mixed Hodge structure on the hypercohomology spaces

$$\mathbb{H}^i(E(J), v_J^* R\psi_f(\mathbb{Z}))$$

We will follow the approach in [23, § 12.1]. We briefly recall the definition of the complex component  $v_J^* \psi_f^H(\mathbb{C})$  of the localized limit integral mixed Hodge complex of sheaves  $v_J^* \psi_f^H$  on  $E(J)$ , in order to fix notations (the notations we adopt here differ slightly from the ones in [23, § 12.1]).

For any non-empty subset  $L$  of  $J$ , we denote by  $\mathcal{I}_{E_L}$  the defining ideal sheaf of  $E_L = \cap_{i \in L} E_i$  in  $X$ . Then  $\mathcal{I}_{E_L} \Omega_X^\bullet(\log E)$  is a subcomplex of  $\Omega_X^\bullet(\log E)$ , and we consider the bifiltered complex

$$\Omega_X^\bullet(\log E)|_{E_L} = \Omega_X^\bullet(\log E) / \mathcal{I}_{E_L} \Omega_X^\bullet(\log E)$$

in  $C^+(E, \mathbb{C})$ , endowed with the quotient filtrations  $W$  and  $F$  (the quotient of the weight filtration, resp. of the stupid filtration on  $\Omega_X^\bullet(\log E)$ ). We define a double complex  $A_L^{\bullet\bullet}$  by

$$A_L^{p,q} = \Omega_X^{p+q+1}(\log E)|_{E_L} / W_p(\Omega_X^{p+q+1}(\log E)|_{E_L})$$

for  $p, q \geq 0$  and we consider  $A_L^{\bullet\bullet}$  as a sheaf on  $E(J)$ . The differentials  $d'_L$  and  $d''_L$  are given by

$$\begin{aligned} d'_L & : A_L^{p,q} \rightarrow A_L^{p+1,q} : \omega \mapsto (dt/t) \wedge \omega \\ d''_L & : A_L^{p,q} \rightarrow A_L^{p,q+1} : \omega \mapsto d\omega \end{aligned}$$

We put an increasing filtration  $W(M)$  (the *monodromy weight filtration*) on  $A_L^{\bullet\bullet}$  by defining  $W(M)_r A_L^{p,q}$  as the image of  $W_{r+2p+1} \Omega_X^{p+q+1}(\log E)$  in  $A_L^{p,q}$ ; this induces a filtration  $W(M)$  on the associated simple complex  $C_L^\bullet = s(A_L^{\bullet\bullet})$ . We also endow  $C_L^\bullet$  with a decreasing filtration  $F$  by putting  $F^r C_L^\bullet = \oplus_{p+q=n, q \geq r} A_L^{p,q}$ .

When  $L$  varies over the non-empty subsets of  $J$ , the bifiltered complexes  $(C_L^\bullet, W(M), F)$  form a  $J$ -cocubical complex of bifiltered objects in  $C^+(E(J), \mathbb{C})$ , whose associated simple complex  $s_\square(C_L^\bullet, W(M), F)$  is denoted by  $v_J^* \psi_f^H(\mathbb{C})$ . The complex  $s_\square(C_L^\bullet)$  is quasi-isomorphic to  $v_J^* R\psi_f(\mathbb{C})$ .

**5.3. The specialization map on the mixed Hodge level.** We keep the notations of Section 5.2. We would like to lift the natural specialization map  $\mathbb{C}_{E(J)} \rightarrow v_J^* R\psi_f(\mathbb{C})$  to a map of bifiltered complexes

$$Hdg^\bullet(E(J))_{\mathbb{C}} \rightarrow v_J^* \psi_f^H(\mathbb{C})$$

(here  $Hdg^\bullet(E(J))_{\mathbb{C}}$  is the complex introduced in Example 5.1). In the global case  $J = I$ , this was done in [23, § 11.3.1].

For any non-empty subset  $M$  of  $I$ , we denote by  $a_M$  the closed immersion  $E_M \rightarrow E$ . In the proof of [23, Thm. 6.12], the following property was shown.

**Lemma 5.4.** *There is a commutative diagram*

$$\begin{array}{ccc} Gr_m^W(\Omega_X^\bullet(\log E)) & \xrightarrow{\cong} & \bigoplus_{\substack{M \subset I \\ |M|=m}} (a_M)_* \Omega_{E_M}^\bullet[-m] \\ \downarrow & & \downarrow \\ Gr_m^W(\Omega_X^\bullet(\log E)|_{E_L}) & \xrightarrow{\cong} & \bigoplus_{\substack{M \subset I \\ |M|=m}} (a_{L \cup M})_* \Omega_{E_{L \cup M}}^\bullet[-m] \end{array}$$

for any couple non-empty subset  $L$  of  $I$  and any integer  $m \geq 1$ . Here the upper horizontal map is the isomorphism induced by the residue map, the left vertical arrow comes from the natural projection  $\Omega_X^\bullet(\log E) \rightarrow \Omega_X^\bullet(\log E)|_{E_L}$ , and the right vertical arrow is the obvious restriction map.

By Example 5.3 (applied to  $X = E_L$  and the cover  $\{E_{L \cup i} \mid i \in I\}$ ), we have for each non-empty subset  $L$  of  $J$  a quasi-isomorphism

$$(a_L)_* \Omega_{E_L}^\bullet \cong \bigoplus_{p+q=\bullet, p \geq 0} \bigoplus_{M \subset I, |M|=p+1} (a_{L \cup M})_* \Omega_{E_{L \cup M}}^q$$

and using the isomorphism in Lemma 5.4, we get a natural quasi-isomorphism

$$(a_L)_* \Omega_{E_L}^\bullet \rightarrow \bigoplus_{p+q=\bullet, p \geq 0} Gr_{p+1}^W(\Omega_X^{p+q+1}(\log E)|_{E_L})$$

whence a morphism of  $J$ -cocubical systems of bifiltered complexes

$$(5.1) \quad \sigma_L : ((a_L)_* \Omega_{E_L}^\bullet, W, F) \rightarrow (C_L^\bullet, W(M), F)$$

where the left hand side is defined as in Example 5.1. Passing to the associated simple complexes, we get a map of bifiltered complexes

$$(5.2) \quad \sigma : Hdg^\bullet(E(J))_{\mathbb{C}} \rightarrow v_J^* \psi_f^H(\mathbb{C})$$

**Lemma 5.5.** *The square*

$$\begin{array}{ccc} \mathbb{C}_{E(J)} & \xrightarrow{spec} & v_J^* R\psi_f(\mathbb{C}) \\ \downarrow & & \downarrow \\ Hdg^\bullet(E(J))_{\mathbb{C}} & \xrightarrow{\sigma} & v_J^* \psi_f^H(\mathbb{C}) \end{array}$$

commutes in  $D^+(E(J), \mathbb{C})$ . The upper horizontal map is the natural specialization map, and the vertical maps are the natural comparison isomorphisms in  $D^+(E(J), \mathbb{C})$  which are part of the mixed Hodge complexes  $Hdg^\bullet(E(J))$ , resp.  $v_J^* \psi_f^H$ .

*Proof.* In the global case  $I = J$ , Peters and Steenbrink defined a limit integral mixed Hodge complex of sheaves  $\psi_f^H$  on  $E$  (see [23, 11.2.7]); we denote by  $\psi_f^H(\mathbb{C})$  its complex component. The complex  $\psi_f^H(\mathbb{C})$  is defined as the simple complex associated to the double complex

$$A^{p,q} = \Omega^{p+q+1}(\log E)/W_p \Omega^{p+q+1}(\log E)$$

where the differentials are defined in the same way as for  $A_L^{\bullet\bullet}$ . By [23, 11.3.1], we have the following commutative diagram in  $D^+(E, \mathbb{C})$ :

$$\begin{array}{ccc} \mathbb{C}_E & \xrightarrow{\text{spec}} & R\psi_f(\mathbb{C}) \\ \downarrow & & \downarrow \\ \text{Hdg}^\bullet(E)_{\mathbb{C}} & \xrightarrow{\beta} & \psi_f^H(\mathbb{C}) \end{array}$$

The vertical maps are the natural comparison isomorphisms, and  $\beta$  is constructed by means of the residue maps:

$$\bigoplus_{|L|=p+1} (a_L)_* \Omega_{E_L}^q \cong Gr_{p+1}^W \Omega^{p+q+1}(\log E) \rightarrow A^{p,q} = \Omega^{p+q+1}(\log E) / W_p(\Omega^{p+q+1}(\log E))$$

Applying the functor  $a_L^* : D^+(E, \mathbb{C}) \rightarrow D^+(E_L, \mathbb{C})$ , for any non-empty subset  $L$  of  $J$ , we get a commutative diagram in  $D^+(E_L, \mathbb{C})$ :

$$\begin{array}{ccc} \mathbb{C}_{E_L} & \xrightarrow{\text{spec}} & a_L^* R\psi_f(\mathbb{C}) \\ \downarrow & & \downarrow \\ a_L^*(\text{Hdg}^\bullet(E)_{\mathbb{C}}) & \xrightarrow{\beta_L} & a_L^* \psi_f^H(\mathbb{C}) \end{array}$$

We denote by  $b_L$  the closed immersion  $E_L \rightarrow E(J)$ , and we define an isomorphism  $\gamma_L : a_L^*(\text{Hdg}^\bullet(E)_{\mathbb{C}}) \rightarrow \Omega_{E_L}^\bullet$  in  $D^+(E_L, \mathbb{C})$  as the composition of the natural isomorphisms

$$a_L^*(\text{Hdg}^\bullet(E)_{\mathbb{C}}) \cong a_L^* \mathbb{C}_E \cong \mathbb{C}_{E_L} \cong \Omega_{E_L}^\bullet$$

in  $D^+(E_L, \mathbb{C})$ .

In view of Example 5.2, it suffices to prove the following claim: *there exists a commutative diagram*

$$\begin{array}{ccc} a_L^*(\text{Hdg}^\bullet(E)_{\mathbb{C}}) & \xrightarrow{\beta_L} & a_L^* \psi_f^H(\mathbb{C}) \\ \gamma_L \downarrow & & \downarrow \delta_L \\ \Omega_{E_L}^\bullet & \xrightarrow{\sigma_L} & a_L^* C_L^\bullet \end{array}$$

in  $D^+(E_L, \mathbb{C})$ , where the vertical arrows are isomorphisms,  $\sigma_L$  is the map in (5.1), and  $\delta_L$  is a morphism in  $C^+(E_L, \mathbb{C})$  such that the composition

$$v_J^* R\psi_f(\mathbb{C}) \cong s_\square(((b_L)_* a_L^* \psi_f^H(\mathbb{C}))_{L \in \square_J}) \xrightarrow{s_\square(\delta_L)} s_\square((C_L^\bullet)_{L \in \square_J}) = v_J^* \psi_f^H(\mathbb{C})$$

coincides with the natural comparison isomorphism  $v_J^* R\psi_f(\mathbb{C}) \cong v_J^* \psi_f^H(\mathbb{C})$  in  $D^+(E(J), \mathbb{C})$ .

Consider the natural map

$$\delta'_L : a_L^*(\Omega^\bullet(\log E) / W_0 \Omega^\bullet(\log E)) \rightarrow a_L^*(\Omega^\bullet(\log E)|_{E_J} / W_0 \Omega^\bullet(\log E)|_{E_J})$$

in  $C^+(E_L, \mathbb{C})$ . Applying the functor  $a_L^*$  to the diagram in Lemma 5.4, we see immediately that  $\delta'_L$  is a filtered quasi-isomorphism w.r.t. the quotient of the weight filtrations. The map  $\delta'_L$  defines a morphism of double complexes  $a_L^* A^{p,q} \rightarrow a_L^* A_L^{p,q}$ , inducing a map  $\delta_L$  on the associated simple complexes. Since  $\delta'_L$  is a filtered quasi-isomorphism,  $\delta_L$  is a filtered quasi-isomorphism w.r.t. the second filtration on the double complexes. The fact that  $\sigma_L \circ \gamma_L = \delta_L \circ \beta_L$  follows from Lemma 5.4.  $\square$

**Proposition 5.6.** *The natural specialization map  $\mathbb{Z} \rightarrow R\psi_f(\mathbb{Z})$  induces canonical isomorphisms*

$$Gr_F^0 H^i(E(J), \mathbb{C}) \cong Gr_F^0 \mathbb{H}^i(E(J), v_J^* R\psi_f(\mathbb{C}))$$

for  $i \geq 0$ . In particular, by restricting to the weight zero part, we get canonical isomorphisms

$$W_0 H^i(E(J), \mathbb{Q}) \cong W_0 \mathbb{H}^i(E(J), v_J^* R\psi_f(\mathbb{Q}))$$

*Proof.* It suffices to prove the statement after replacing rational coefficients by complex coefficients. Then the second isomorphism follows from the first since both sides have Hodge type  $(0, 0)$ .

Applying the functor  $Gr_F^0$  to the morphism  $\sigma_L$  in (5.1), we get a morphism of complexes

$$Gr_F^0(\sigma_L) : (a_L)_* \mathcal{O}_{E_L} \rightarrow \Omega_X^{\bullet+1}(\log E)|_{E_L} / W_\bullet(\Omega_X^{\bullet+1}(\log E)|_{E_L}) = Gr_{\bullet+1}^W(\Omega_X^{\bullet+1}(\log E)|_{E_L})$$

and this map is a quasi-isomorphism: by Lemma 5.4, the target is isomorphic to the complex  $\oplus_{M \subset I, |M|=\bullet+1} (a_{L \cup M})_* \mathcal{O}_{E_{L \cup M}}$ . Passing to the associated simple complexes, we see that

$$Gr_F^0(\sigma) : Gr_F^0 Hdg^\bullet(E(J))_{\mathbb{C}} \rightarrow Gr_F^0 \psi_f^H(\mathbb{C})$$

is a quasi-isomorphism, as well.  $\square$

#### 5.4. Mixed Hodge structure on the nearby and vanishing cohomology.

Consider a complex variety  $X$  and a flat morphism  $f : X \rightarrow \text{Spec } \mathbb{C}[t]$  with smooth generic fiber. Let  $x$  be any complex point of  $X_s$ . The cohomology spaces  $R^i \psi_f(\mathbb{Z})_x$  carry a natural mixed Hodge structure [17, 15.13][23, § 12.1.2]. Take a strictly semi-stable model  $(Y, g, \varphi)$  of  $(f, x)$  (Definition 4.7), with  $Y_s = \sum_{i \in I} E_i$ ; then  $\varphi^{-1}(x) = E(J)$  for some non-empty subset  $J$  of  $I$ . There are canonical isomorphisms

$$\mathbb{H}^i(E(J), v_J^* R\psi_g(\mathbb{Z})) \cong R^i \psi_f(\mathbb{Z})_x$$

for  $i \geq 0$ , and in this way  $R^i \psi_f(\mathbb{Z})_x$  inherits an integral mixed Hodge structure, which does not depend on the chosen strictly semi-stable model.

If we denote by  $R\Theta_f$  the vanishing cycles functor, then  $R^i \Theta_f(\mathbb{Z})_x$  also carries a natural mixed Hodge structure [18, 1.1]. We have

$$R^i \Theta_f(\mathbb{Z})_x \cong R^i \psi_f(\mathbb{Z})_x$$

for  $i > 0$ , so  $R^i \Theta_f(\mathbb{Z})_x$  inherits a mixed Hodge structure. For  $i = 0$ ,  $R^i \Theta_f(\mathbb{Z})_x$  carries a pure Hodge structure of weight zero.

If  $X$  is smooth at  $x$ , then  $R^i \psi_f(\mathbb{Z})_x$  is isomorphic to the degree  $i$  integral singular cohomology of the topological Milnor fiber of  $f$  at  $x$  for each  $i$ , so this cohomology carries a mixed Hodge structure. If  $x$  is an isolated singularity of  $f$ , then this mixed Hodge structure was constructed in [25]. Likewise,  $R^i \Theta_f(\mathbb{Z})_x$  is isomorphic to the degree  $i$  reduced integral singular cohomology of the topological Milnor fiber of  $f$  at  $x$ .

**5.5. Non-archimedean interpretation of the weight zero subspace.** In this section, we put  $R = \mathbb{C}[[t]]$  and  $K = \mathbb{C}((t))$ , and we endow  $K$  with the absolute value  $|\cdot|_r$  for some fixed  $r \in ]0, 1[$ .

**Theorem 5.7.** *Let  $X$  be a complex variety, endowed with a flat morphism  $f : X \rightarrow \text{Spec } \mathbb{C}[t]$  with smooth generic fiber. Let  $x$  be a point of  $X(\mathbb{C})$  with  $f(x) = 0$ . Denote by  $\mathcal{F}_x$  the analytic Milnor fiber of  $f$  at  $x$ .*

For each  $i \geq 0$ , there exists canonical isomorphisms

$$\begin{aligned} \alpha & : H^i(\overline{\mathcal{F}}_x, \mathbb{Q}) \rightarrow W_0 R^i \psi_f(\mathbb{Q})_x \\ \alpha' & : \widetilde{H}^i(\overline{\mathcal{F}}_x, \mathbb{Q}) \rightarrow W_0 R^i \Theta_f(\mathbb{Q})_x \end{aligned}$$

*Proof.* Take a strictly semi-stable model  $(Y, g, \varphi)$  of  $(f, x)$  (Definition 4.7), denote by  $d$  its ramification index, and by  $\mathcal{Y}$  the  $t$ -adic completion of  $Y$ . Put  $E = \varphi^{-1}(x)$ , and denote by  $v : E \rightarrow \mathcal{Y}_s$  the closed immersion. Then  $\varphi$  induces an isomorphism of  $(K_r)^a$ -analytic spaces

$$\overline{]E[} := sp_{\mathcal{Y}(r)}^{-1}(E) \widehat{\times}_{K_r(d)} \widehat{(K_r)^a} \cong \overline{\mathcal{F}}_x$$

and an isomorphism of mixed Hodge structures

$$R^i \psi_f(\mathbb{Q})_x \cong \mathbb{H}^i(E, v^* R\psi_g(\mathbb{Q}))$$

On the other hand, Berkovich proved in [8, Thm 1.1(c)] that there exists a canonical isomorphism  $H^i(E^{an}, \mathbb{Q}) \cong W_0 H^i(E, \mathbb{Q})$ , and it follows from Proposition 4.4 and Lemma 4.5 that there exists a canonical isomorphism  $H^i(E^{an}, \mathbb{Q}) \cong H^i(\overline{]E[}, \mathbb{Q})$ .

Hence, we obtain an isomorphism  $H^i(\overline{\mathcal{F}}_x, \mathbb{Q}) \cong W_0 H^i(E, \mathbb{Q})$  and, by Proposition 5.6, an isomorphism

$$H^i(\overline{\mathcal{F}}_x, \mathbb{Q}) \cong W_0 R^i \psi_f(\mathbb{Q})_x$$

A standard argument shows that this isomorphism does not depend on the chosen semi-stable model.

The mixed Hodge structure on  $R^0 \psi_f(\mathbb{Q})_x$  is pure of weight zero, so for  $i = 0$  the map  $\alpha$  is an isomorphism

$$\alpha : H^0(\overline{\mathcal{F}}_x, \mathbb{Q}) \rightarrow R^0 \psi_f(\mathbb{Q})_x$$

Passing to reduced cohomology yields the natural isomorphism  $\alpha'$ .  $\square$

**Remark.** Using the same methods as in [3] one can generalize Theorem 5.7 as follows: if  $f : X \rightarrow \text{Spec } \mathbb{C}[t]$  is a flat morphism of complex varieties,  $Z$  a proper subvariety of the special fiber, and  $\mathfrak{Z}$  the formal completion of  $f$  along  $Z$ , then there is for each  $i \geq 0$  a canonical isomorphism

$$\alpha : H^i(\overline{\mathfrak{Z}(r)}_\eta, \mathbb{Q}) \rightarrow W_0 \mathbb{H}^i(Z, R\psi_f(\mathbb{Q}))$$

One reduces to the case where  $f$  has smooth generic fiber by taking a hypercovering of  $f$ ; then one uses the proof of Theorem 5.7.  $\square$

**Corollary 5.8.** *If  $X$  is smooth, of pure dimension  $n + 1$ , and if we denote by  $s$  the dimension of the singular locus of  $f$  at  $x$ , then  $H^i(\overline{\mathcal{F}}_x, \mathbb{Q}) = 0$  for  $i \notin \{0, n - s, n - s + 1, \dots, n\}$ . If  $s < n$ , then  $\overline{\mathcal{F}}_x$  is arc-connected.*

*Proof.* It is well-known that the reduced cohomology of the topological Milnor fiber of  $f$  at  $x$  vanishes for  $i \notin \{n - s, n - s + 1, \dots, n\}$ . If  $s < n$ , then  $R^0 \psi_f(\mathbb{Q})_x \cong \mathbb{Q}$ , so  $\overline{\mathcal{F}}_x$  is connected and hence arc-connected by [4, 3.2.1].  $\square$

**Corollary 5.9.** *We assume that  $X$  is smooth of pure dimension  $n + 1$  with  $n > 0$ ,  $f$  is projective and  $x$  is the only singular point of the special fiber  $X_s = f^{-1}(0)$ . We*



denote by  $\mathcal{X}$  the  $t$ -adic completion of  $f$ . Then we have an isomorphism of long exact sequences

$$\begin{array}{ccccccc} \rightarrow H^i((X_s)^{an}, \mathbb{Q}) & \rightarrow & H^i(\overline{\mathcal{X}(r)}_\eta, \mathbb{Q}) & \rightarrow & \tilde{H}^i(\overline{\mathcal{X}}_x, \mathbb{Q}) & \rightarrow & H^{i+1}((X_s)^{an}, \mathbb{Q}) \rightarrow \\ \gamma \downarrow \cong & & \beta \downarrow \cong & & \cong \downarrow \alpha' & & \downarrow \cong \\ \rightarrow W_0 H^i(X_s, \mathbb{Q}) & \rightarrow & W_0 \mathbb{H}^i(X_s, R\psi_f(\mathbb{Q})) & \rightarrow & W_0 R^i \Theta_f(\mathbb{Q})_x & \rightarrow & W_0 H^{i+1}(X_s, \mathbb{Q}) \rightarrow \end{array}$$

where the upper long exact sequence is the one from Proposition 4.6, the lower one is obtained by applying the exact functor  $Gr_0^W$  to the canonical long exact sequence of mixed Hodge structures [18, 1.1]

$$\dots \rightarrow H^i(X_s, \mathbb{Q}) \rightarrow \mathbb{H}^i(X_s, R\psi_f(\mathbb{Q})) \rightarrow \mathbb{H}^i(X_s, R\Theta_f(\mathbb{Q})) \cong R^i \Theta_f(\mathbb{Q})_x \rightarrow \dots$$

and where  $\beta$  is the isomorphism of [3, 5.1] and  $\gamma$  is the isomorphism of [8, Thm 1.1(c)].

This diagram breaks up in commutative squares of isomorphisms, for each  $0 \leq i < n$ ,

$$\begin{array}{ccc} H^i((X_s)^{an}, \mathbb{Q}) & \xrightarrow{\sim} & H^i(\overline{\mathcal{X}(r)}_\eta, \mathbb{Q}) \\ \gamma \downarrow \cong & & \cong \downarrow \beta \\ W_0 H^i(X_s, \mathbb{Q}) & \xrightarrow{\sim} & W_0 \mathbb{H}^i(X_s, R\psi_f(\mathbb{Q})) \end{array}$$

as well as an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow H^n((X_s)^{an}, \mathbb{Q}) & \rightarrow & H^n(\overline{\mathcal{X}(r)}_\eta, \mathbb{Q}) & \rightarrow & H^n(\overline{\mathcal{X}}_x, \mathbb{Q}) & \rightarrow & H^{n+1}((X_s)^{an}, \mathbb{Q}) \rightarrow 0 \\ \gamma \downarrow \cong & & \beta \downarrow \cong & & \cong \downarrow \alpha & & \downarrow \cong \\ 0 \rightarrow W_0 H^n(X_s, \mathbb{Q}) & \rightarrow & W_0 \mathbb{H}^n(X_s, R\psi_f(\mathbb{Q})) & \rightarrow & W_0 R^n \psi_f(\mathbb{Q})_x & \rightarrow & W_0 H^{n+1}(X_s, \mathbb{Q}) \rightarrow 0 \end{array}$$

It is well-known that any analytic germ of an isolated hypersurface singularity can be embedded in a projective morphism  $f$  as in Corollary 5.9; see [10, § 1.1].

## 6. COMPARISON WITH THE MOTIVIC MILNOR FIBER

In this final section, we consider the motivic counterparts of the above results. Let  $k$  be a field of characteristic zero, and put  $R = k[[t]]$  and  $K = k((t))$ . We fix an element  $r$  in  $]0, 1[$  and we endow  $K$  with the  $t$ -adic absolute value  $|\cdot|_r$ . Since  $r$  will remain fixed throughout this section, we simplify notation by writing  $K$  for the non-archimedean field  $K_r$  and  $\mathcal{X}_\eta$  for the generic fiber  $\mathcal{X}(r)_\eta$  of a special formal  $R$ -scheme in the category of  $K_r$ -analytic spaces. For each integer  $d > 0$ , we put  $K(d) = K[t_d]/((t_d)^d - t)$  and we denote by  $R(d)$  the normalization of  $R$  in  $K(d)$ .

**6.1. Motivic volume of a formal scheme.** Let  $\mathcal{X}$  be a generically smooth special formal  $R$ -scheme. In [19, 7.39] we defined the motivic volume  $S(\mathcal{X}; \widehat{K}^s)$  of  $\mathcal{X}$ . It is an element of the localized Grothendieck ring of  $\mathcal{X}_0$ -varieties  $\mathcal{M}_{\mathcal{X}_0}$  (see for instance [21, § 3.1] for the definition of  $\mathcal{M}_{\mathcal{X}_0}$ ). The aim of the current section is to explain the behaviour of the motivic volume under extension of the base ring  $R$ . This result will be used further on to give an expression of the motivic volume in terms of a semi-stable model of  $\mathcal{X}$  (Theorem 6.11).

As we noted in [19, 7.40], the motivic volume depends in general on the choice of the uniformizer  $t$ , or, more precisely, on the  $K$ -fields  $K(d)$ . If  $k$  is algebraically closed, then up to  $K$ -isomorphism,  $K(d)$  is the unique extension of  $K$  of degree  $d$ , and the motivic volume is independent of the choice of uniformizer.

In order to speak about the motivic volume of a special formal  $R(d)$ -scheme when  $k$  is not algebraically closed, we fix the uniformizer  $t_d$  in  $R(d)$ . This yields an isomorphism of  $k$ -algebras  $R(d) \cong k[[t_d]]$  and natural isomorphisms of  $R$ -algebras  $(R(d))(e) \cong R(d \cdot e)$  for  $d, e > 0$ .

As usual, we denote by  $\mathbb{L}$  the class of the affine line  $\mathbb{A}_{\mathcal{X}_0}^1$  in  $\mathcal{M}_{\mathcal{X}_0}$ . This is a unit in  $\mathcal{M}_{\mathcal{X}_0}$ . We'll denote by  $\mathcal{R}_{\mathcal{X}_0}$  the subring

$$\mathcal{R}_{\mathcal{X}_0} := \mathcal{M}_{\mathcal{X}_0} \left[ \frac{T^b}{T^b - \mathbb{L}^a} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{N}^*}$$

of  $\mathcal{M}_{\mathcal{X}_0}[[T]]$ , and by  $\mathcal{R}'_{\mathcal{X}_0}$  the subring of  $\mathcal{M}_{\mathcal{X}_0}[[T]]$  consisting of elements of the form  $P(T)/Q(T)$ , with  $P(T), Q(T)$  polynomials over  $\mathcal{M}_{\mathcal{X}_0}$  such that  $Q(0)$  is a unit in  $\mathcal{M}_{\mathcal{X}_0}$ ,  $Q(T)$  is monic, and the degree of  $Q(T)$  is at least the degree of  $P(T)$ . The ring  $\mathcal{R}_{\mathcal{X}_0}$  contains  $\mathcal{R}'_{\mathcal{X}_0}$ . There exists a unique morphism of  $\mathcal{M}_{\mathcal{X}_0}$ -algebras

$$\lim_{T \rightarrow \infty} : \mathcal{R}'_{\mathcal{X}_0} \rightarrow \mathcal{M}_{\mathcal{X}_0}$$

mapping  $P(T)/Q(T)$  (with  $P(T), Q(T)$  as above) to the coefficient of  $T^{\deg Q}$  in  $P(T)$ , where  $\deg Q$  denotes the degree of  $Q(T)$ . It restricts to the morphism

$$\lim_{T \rightarrow \infty} : \mathcal{R}_{\mathcal{X}_0} \rightarrow \mathcal{M}_{\mathcal{X}_0}$$

from [19, 7.35].

For any ring  $A$ , any element  $a(T) = \sum_{i \geq 0} a_i T^i$  of  $A[[T]]$ , and any integer  $d > 0$ , we put  $a(T)[d] = \sum_{i \geq 0, d|i} a_i T^i \in A[[T]]$ .

**Lemma 6.1.** *For any  $(p, q, r) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}^*$  with  $q \leq r$ ,*

$$\frac{T^q}{T^r - \mathbb{L}^p} \in \mathcal{R}_{\mathcal{X}_0}$$

*Proof.* For any  $(p, q, r) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}^*$  we use the notation

$$D_{q,r}^p := \frac{T^q}{T^r - \mathbb{L}^p} \in \mathcal{M}_{\mathcal{X}_0}[[T]]$$

By definition,  $D_{r,r}^p \in \mathcal{R}_{\mathcal{X}_0}$ . Moreover, the relation  $D_{0,r}^p = \mathbb{L}^{-p} (D_{r,r}^p - 1)$  shows that  $D_{0,r}^p \in \mathcal{R}_{\mathcal{X}_0}$ .

Put  $I = \{(q, r) \in \mathbb{N} \times \mathbb{N}^* \mid q \leq r\}$ . We will show that  $D_{q,r}^p \in \mathcal{R}_{\mathcal{X}_0}$  for any  $(p, q, r) \in \mathbb{Z} \times I$ , by induction on  $(q, r)$  w.r.t. the lexicographic ordering on  $I$ . We may assume that  $0 < q < r$ . We have

$$\begin{aligned} D_{q,r}^p &= \frac{T^r}{T^r - \mathbb{L}^p} \cdot \frac{1}{T^{r-q} - 1} - \frac{1}{T^r - \mathbb{L}^p} \cdot \frac{T^q}{T^{r-q} - 1} \\ &\equiv -D_{0,r}^p \cdot D_{q,r-q}^0 \pmod{\mathcal{R}_{\mathcal{X}_0}} \end{aligned}$$

We know that  $D_{0,r}^p \in \mathcal{R}_{\mathcal{X}_0}$ , and if  $q \leq r - q$ , then  $D_{q,r-q}^0 \in \mathcal{R}_{\mathcal{X}_0}$  by the induction hypothesis. Hence, we may assume that  $q > r - q$ . Then we write

$$D_{0,r}^p \cdot D_{q,r-q}^0 = D_{2q-r,r}^p \cdot D_{r-q,r-q}^0$$

and since  $0 < 2q - r < q$  the induction hypothesis implies that  $D_{2q-r,r}^p$  belongs to  $\mathcal{R}_{\mathcal{X}_0}$ .  $\square$

**Lemma 6.2.** *For any integer  $d > 0$ , the morphism of  $\mathcal{M}_{\mathcal{X}_0}$ -modules*

$$\phi_d : \mathcal{M}_{\mathcal{X}_0}[[T]] \rightarrow \mathcal{M}_{\mathcal{X}_0}[[T]] : a(T) \mapsto a(T)[d]$$

*restricts to a  $\mathcal{M}_{\mathcal{X}_0}$ -module endomorphism of  $\mathcal{R}_{\mathcal{X}_0}$ . Moreover,*

$$\lim_{T \rightarrow \infty} \circ \phi_d = \lim_{T \rightarrow \infty}$$

*Proof.* Consider an integer  $n > 0$ , a tuple  $a \in \mathbb{Z}^n$ , and a tuple  $b \in (\mathbb{N}^*)^n$ . We put

$$C_{a,b} := \left( \prod_{i=1}^n \frac{T^{b_i}}{T^{b_i} - \mathbb{L}^{a_i}} \right) [d] \in \mathcal{M}_{\mathcal{X}_0}[[T]]$$

It suffices to show that  $C_{a,b} \in \mathcal{R}_{\mathcal{X}_0}$  and that

$$\lim_{T \rightarrow \infty} C_{a,b} = 1$$

For any pair of tuples  $u, v \in \mathbb{Z}^n$ , we put  $u \cdot v = \sum_{i=1}^n u_i \cdot v_i$ . For each  $i \in \{1, \dots, n\}$ , we put  $m_i = \text{lcm}(b_i, d)$  and  $e_i = m_i/b_i$ . If we put

$$S = \{u \in \mathbb{N}^n \mid 1 \leq u_i \leq e_i \text{ for all } i, \text{ and } d \mid u \cdot b\}$$

then the map

$$S \times \mathbb{N}^n \rightarrow \{w \in (\mathbb{N}^*)^n \mid w \cdot b \in d\mathbb{N}\} : (u, v) \mapsto (u_1 + v_1 e_1, \dots, u_n + v_n e_n)$$

is a bijection. Therefore

$$C_{a,b} = \left( \sum_{u \in S} \mathbb{L}^{-a \cdot u} T^{b \cdot u} \right) \cdot \left( \prod_{i=1}^n \frac{\mathbb{L}^{a_i e_i}}{T^{b_i e_i} - \mathbb{L}^{a_i e_i}} \right)$$

Since  $e = (e_1, \dots, e_n)$  belongs to  $S$ , the first factor is a polynomial with leading term  $\mathbb{L}^{-a \cdot e} T^{b \cdot e}$ , so we see that  $C_{a,b}$  belongs to  $\mathcal{R}'_{\mathcal{X}_0}$  and

$$\lim_{T \rightarrow \infty} C_{a,b} = 1$$

as required. Lemma 6.1 shows that  $C_{a,b} \in \mathcal{R}_{\mathcal{X}_0}$ .  $\square$

**Proposition 6.3.** *For any generically smooth special formal  $R$ -scheme  $\mathcal{X}$  and any integer  $d > 0$ ,*

$$S(\mathcal{X}; \widehat{K^s}) = S(\mathcal{X} \times_R R(d); \widehat{K(d)^s})$$

*in  $\mathcal{M}_{\mathcal{X}_0}$ .*

*Proof.* By [19, 7.38+39] we may assume that  $\mathcal{X}_\eta$  admits a  $\mathcal{X}$ -bounded gauge form  $\omega$  (in the sense of [19, 2.11]). For each  $n > 0$  we denote by  $\omega(n)$  the pull-back of  $\omega$  to  $\mathcal{X}_\eta \times_K K(n)$ . This is a  $\mathcal{X} \times_R R(n)$ -bounded gauge form.

In [19, 4.9] we defined the volume Poincaré series  $S(\mathcal{X}, \omega; T)$  of  $(\mathcal{X}, \omega)$  by

$$S(\mathcal{X}, \omega; T) = \sum_{n > 0} \left( \int_{\mathcal{X} \times_R R(n)} |\omega(n)| \right) T^n \in \mathcal{M}_{\mathcal{X}_0}[[T]]$$

(the coefficients are motivic integrals). It is clear from the definition (and from our choice of uniformizer in  $R(d)$ ) that

$$S(\mathcal{X} \times_R R(d), \omega(d); T) = S(\mathcal{X}, \omega; T)[d]$$

We showed in [19, 7.14] that  $S(\mathcal{X}, \omega; T)$  is rational; more precisely, it belongs to  $\mathcal{R}_{\mathcal{X}_0}$ . By definition,

$$S(\mathcal{X}; \widehat{K^s}) = - \lim_{T \rightarrow \infty} S(\mathcal{X}, \omega; T)$$

By Lemma 6.2,

$$\begin{aligned}
S(\mathcal{X}; \widehat{K^s}) &= - \lim_{T \rightarrow \infty} S(\mathcal{X}, \omega; T) \\
&= - \lim_{T \rightarrow \infty} S(\mathcal{X} \times_R R(d), \omega(d); T) \\
&= S(\mathcal{X} \times_R R(d); \widehat{K(d)^s})
\end{aligned}$$

□

**Proposition 6.4.** *Let  $k'$  be a field containing  $k$ , and put  $R' = k'[[t]]$ . Let  $\mathcal{X}$  be a generically smooth special formal  $R$ -scheme, and put  $\mathcal{X}' = \mathcal{X} \widehat{\times}_R R'$ . Then  $S(\mathcal{X}'; \widehat{K^s})$  is the image of  $S(\mathcal{X}; \widehat{K^s})$  under the base change morphism  $\mathcal{M}_{\mathcal{X}_0} \rightarrow \mathcal{M}_{\mathcal{X}'_0}$ .*

*Proof.* This is obvious from the definition of the motivic volume. □

**6.2. Motivic volume of a rigid variety.** Let  $\mathcal{X}$  be a generically smooth special formal  $R$ -scheme, and assume that  $\mathcal{X}$  is *stft* or that  $\mathcal{X}_\eta$  admits a universally bounded gauge form in the sense of [19, 7.41]. Such a universally bounded gauge form exists, in particular, if  $\mathcal{X}$  is the formal spectrum of a regular local  $R$ -algebra [19, 7.23+42].

In [21, 8.3] and [19, 7.43] we defined the motivic volume  $S(\mathcal{X}_\eta; \widehat{K^s})$  of  $\mathcal{X}_\eta$  as the image of  $S(\mathcal{X}; \widehat{K^s})$  under the forgetful morphism  $\mathcal{M}_{\mathcal{X}_0} \rightarrow \mathcal{M}_k$  and we showed that this definition only depends on  $\mathcal{X}_\eta$  and not on the model  $\mathcal{X}$ .

**Proposition 6.5.** *Let  $d > 0$  be an integer and  $X$  a separated smooth rigid  $K$ -variety. Assume that  $X$  and  $X \times_K K(d)$  admit universally bounded gauge forms, or that  $X$  is quasi-compact. Then*

$$S(X \times_K K(d); \widehat{K(d)^s}) = S(X; \widehat{K^s})$$

*in  $\mathcal{M}_k$ .*

*Proof.* This follows immediately from Proposition 6.3. □

**Proposition 6.6.** *Let  $k'$  be any field containing  $k$ , and put  $L = k'((t))$ . Let  $X$  be a separated smooth rigid  $K$ -variety. Assume that  $X$  and  $X \times_K L$  admit universally bounded gauge forms, or that  $X$  is quasi-compact. Then  $S(X \times_K L; \widehat{L^s})$  is the image of  $S(X; \widehat{K^s})$  under the base change morphism  $\mathcal{M}_k \rightarrow \mathcal{M}_{k'}$ .*

*Proof.* This follows immediately from Proposition 6.4. □

**Remark.** Even though the valuation on  $\widehat{K^a}$  is not discrete and the construction of the motivic volume does not apply to  $\widehat{K^a}$ -analytic spaces, the above results justify the hope that one can associate a motivic volume  $Vol(X)$  to separated smooth quasi-compact rigid varieties  $X$  over  $\widehat{K^a}$  (or an even larger class of  $\widehat{K^a}$ -analytic spaces). This volume should have the property that  $Vol(Y \widehat{\times}_K \widehat{K^a}) = S(Y; \widehat{K^s})$  when  $Y$  is a separated smooth quasi-compact rigid  $K$ -variety, and opens the way to a theory of motivic integration on  $\widehat{K^a}$ -analytic spaces. □

**6.3. Motivic Milnor fiber.** We start with an auxiliary result.

**Lemma 6.7.** *For any morphism of generically smooth special formal  $R$ -schemes  $h : \mathcal{Y} \rightarrow \mathcal{X}$  such that  $h_\eta$  is an isomorphism, we have*

$$S(\mathcal{Y}; \widehat{K}^s) = S(\mathcal{X}; \widehat{K}^s)$$

in  $\mathcal{M}_{\mathcal{X}_0}$ . Here the left hand side is viewed as an element of  $\mathcal{M}_{\mathcal{X}_0}$  via the forgetful morphism  $\mathcal{M}_{\mathcal{X}_0} \rightarrow \mathcal{M}_{\mathcal{X}_0}$  induced by  $h_0$ .

*Proof.* This is straightforward from the definition of the motivic volume in [19, 7.39].  $\square$

Let  $X$  be a  $k$ -variety of pure dimension  $m$ , and let  $f : X \rightarrow \mathbb{A}_k^1 = \text{Spec } k[t]$  be a flat morphism with smooth generic fiber. We fix a closed point  $x$  on the special fiber  $X_s$  of  $f$ . We denote by  $\mathcal{X}$  the  $t$ -adic completion of  $f$ , and by  $\mathcal{X}_x$  the completion of  $f$  at  $x$ , i.e. the special formal  $R$ -scheme  $\text{Spf } \widehat{\mathcal{O}_{X,x}}$ . Then  $(\mathcal{X}_x)_\eta = \mathcal{F}_x$  is the analytic Milnor fiber of  $f$  at  $x$ . We view  $\mathcal{F}_x$  as a  $k'((t))$ -analytic space, with  $k'$  the residue field of  $x$ .

If  $X$  is smooth at  $x$  then  $\mathcal{X}_x$  is the formal spectrum of a regular local  $R$ -algebra, so  $S(\mathcal{F}_x; \widehat{K}^s)$  is defined and equal to  $S(\mathcal{X}_x; \widehat{K}^s) \in \mathcal{M}_x$ . If  $X$  is not smooth at  $x$ , it is not clear if  $\mathcal{F}_x$  admits a universally bounded gauge form. However, we still have the following property.

**Proposition 6.8.** *The analytic Milnor fiber  $\mathcal{F}_x$ , viewed as a  $k'((t))$ -analytic space, uniquely determines the motivic volume*

$$S(\mathcal{X}_x; \widehat{K}^s) \in \mathcal{M}_x$$

*Proof.* The normalization morphism  $\widetilde{\mathcal{X}}_x \rightarrow \mathcal{X}_x$  is a morphism of special formal  $R$ -schemes which induces an isomorphism between the generic fibers because  $\mathcal{F}_x$  is normal (even smooth) and normalization commutes with taking generic fibers [11, 2.1.3]. By Lemma 6.7,

$$S(\widetilde{\mathcal{X}}_x; \widehat{K}^s) = S(\mathcal{X}_x; \widehat{K}^s) \in \mathcal{M}_x$$

Moreover, the  $k'[[t]]$ -algebra of power-bounded analytic functions on  $\mathcal{F}_x$  is canonically isomorphic to  $\mathcal{O}(\widetilde{\mathcal{X}}_x)$  by [12, 7.4.1], so  $\mathcal{F}_x$  determines  $S(\mathcal{X}_x; \widehat{K}^s)$ .  $\square$

**Definition 6.9.** *If  $\mathcal{Y}$  is a generically smooth special formal  $R$ -scheme of pure relative dimension  $d$ , then we define the motivic nearby cycles  $\mathcal{S}_{\mathcal{Y}}$  of  $\mathcal{Y}$  by*

$$\mathcal{S}_{\mathcal{Y}} = \mathbb{L}^d \cdot S(\mathcal{Y}; \widehat{K}^s) \in \mathcal{M}_{\mathcal{Y}_0}$$

Let  $X$  be a  $k$ -variety of pure dimension,  $f : X \rightarrow \text{Spec } k[t]$  a flat morphism with smooth generic fiber, and  $x$  a closed point of the special fiber  $X_s$  of  $f$ . Denote by  $\mathcal{X}/R$  the  $t$ -adic completion of  $f$  and by  $\mathcal{X}_x/R$  the formal completion of  $f$  at  $x$ . We define the motivic Milnor fiber  $\mathcal{S}_{f,x}$  of  $f$  at  $x$  by

$$\mathcal{S}_{f,x} = \mathcal{S}_{\mathcal{X}_x} \in \mathcal{M}_x$$

and the motivic nearby cycles  $\mathcal{S}_f$  of  $f$  by

$$\mathcal{S}_f = \mathcal{S}_{\mathcal{X}} \in \mathcal{M}_{\mathcal{X}_0}$$

If  $X$  is smooth, then Denef and Loeser defined the motivic nearby cycles of  $f$  and the motivic Milnor fiber of  $f$  at  $x$  in [14, §3.5]. By [19, 9.8] these objects coincide with the ones introduced in Definition 6.9. Denef and Loeser showed that if  $k = \mathbb{C}$  and  $X$  is smooth,  $\mathcal{S}_f$  and  $\mathcal{S}_{f,x}$  have the same Hodge realization as the cohomological nearby cycles, resp. Milnor fiber, in an appropriate Grothendieck ring of mixed Hodge modules with monodromy action; see [13, 4.2.1] and [14, 3.5.5].

**6.4. Expression in terms of a semi-stable model.** Now we come to the main result of this section: an expression for the motivic volume of a formal scheme in terms of a semi-stable model. We'll freely use the notation and terminology from [19, §2.5] concerning strict normal crossing divisors on formal schemes.

**Definition 6.10.** *Let  $\mathcal{U}$  be a generically smooth special formal  $R$ -scheme. A strictly semi-stable model  $(\mathcal{V}, g)$  for  $\mathcal{U}$  consists of the following data:*

- (1) *an integer  $d > 0$ , which we call the ramification index of the model  $(\mathcal{V}, g)$ ,*
- (2) *a regular special formal  $R(d)$ -scheme  $\mathcal{V}$ , such that  $\mathcal{V}_s$  is a reduced strict normal crossing divisor on  $\mathcal{V}$ ,*
- (3) *a morphism of special formal  $R$ -schemes  $g : \mathcal{V} \rightarrow \mathcal{U}$  which induces an isomorphism of  $K(d)$ -analytic spaces  $\mathcal{V}_\eta \cong \mathcal{U}_\eta \times_K K(d)$ .*

Such a strictly semi-stable model exists, in particular, when  $\mathcal{U}$  is the formal completion of a generically smooth *stft* formal  $R$ -scheme along a closed subscheme of its special fiber (see the remark following Theorem 4.10).

**Theorem 6.11.** *If  $\mathcal{U}$  is a generically smooth special formal  $R$ -scheme and  $(\mathcal{V}, g)$  is a strictly semi-stable model for  $\mathcal{U}$ , with  $\mathcal{V}_s = \sum_{i \in I} \mathfrak{C}_i$ , then*

$$(6.1) \quad \mathcal{S}_{\mathcal{U}} = \sum_{\emptyset \neq J \subset I} (1 - \mathbb{L})^{|J|-1} [E_J^o] \in \mathcal{M}_{\mathcal{U}_0}$$

*In particular, the right hand side does not depend on the chosen strictly semi-stable model.*

*Proof.* By Proposition 6.3, we may assume that the ramification index  $d$  of the strictly semi-stable model is equal to one, i.e. that  $g_\eta : \mathcal{V}_\eta \rightarrow \mathcal{U}_\eta$  is an isomorphism. Then by Lemma 6.7 we have  $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_{\mathcal{V}}$  in  $\mathcal{M}_{\mathcal{U}_0}$ , and it follows from [19, 7.36] that

$$\mathcal{S}_{\mathcal{V}} = \sum_{\emptyset \neq J \subset I} (1 - \mathbb{L})^{|J|-1} [E_J^o] \in \mathcal{M}_{\mathcal{V}_0}$$

□

**Remark.** If the formal  $R$ -scheme  $\mathcal{V}$  in the statement of Theorem 6.11 is *stft*, then  $E_J^o$  coincides with the stratum  $(\mathcal{V}_s)_J^o$  of the strictly semi-stable  $k$ -scheme  $\mathcal{V}_s$  (Section 2.2) for any  $\emptyset \neq J \subset I = \text{Irr}(\mathcal{V}_s)$ . □

**Corollary 6.12.** *Consider  $f : X \rightarrow \text{Spec } k[t]$ ,  $x$  and  $\mathcal{X}_x$  as in Definition 6.9. If  $(\mathcal{V}, g)$  is a strictly semi-stable model for  $\mathcal{X}_x$ , with  $\mathcal{V}_s = \sum_{i \in I} \mathfrak{C}_i$ , then*

$$(6.2) \quad \mathcal{S}_{f,x} = \sum_{\emptyset \neq J \subset I} (1 - \mathbb{L})^{|J|-1} [E_J^o] \in \mathcal{M}_x$$

*In particular, the right hand side does not depend on the chosen strictly semi-stable model.*

Note in particular that any strictly semi-stable model  $(Y, g, \varphi)$  of the germ  $(f, x)$  (in the sense of Definition 4.7) induces a strictly semi-stable model for  $\mathcal{X}_x$  by taking the formal completion of  $Y$  along  $\varphi^{-1}(x)$  (of course, the projectivity condition can be omitted in Definition 4.7). Therefore, (6.2) also gives an expression for  $\mathcal{S}_{f,x}$  in terms of a semi-stable model of  $(f, x)$ .

As a special case, if  $X$  is smooth, we get an expression for Denef and Loeser's motivic nearby cycles and motivic Milnor fiber in terms of a strictly semi-stable model of  $f$ , resp. of  $(f, x)$ . This expression is not clear from their definition [14, § 3.5].

Moreover, combining Theorem 6.11 with the formula in [19, 7.36] we get an expression for the right hand sides of (6.1) and (6.2) in terms of a resolution of singularities of  $\mathcal{U}$ , resp.  $\mathcal{X}_x$ .

**Remark.** Theorem 6.11 makes it possible to compare our notion of motivic nearby cycles and of motivic volume of a rigid variety with the ones introduced by Ayoub [1][2], after specialization to an appropriate Grothendieck ring of  $k$ -motives. Details will appear elsewhere.  $\square$

Theorem 6.11 is similar in spirit to Theorem 4.10 and the subsequent remark. However, since motivic integrals take their values in a Grothendieck ring, all “non-additive” information is lost when taking the motivic volume  $S(\mathcal{X}_x; \widehat{K^s})$ . As we've seen, non-archimedean geometry provides a powerful additional tool to prove independence results of this type.

## REFERENCES

- [1] J. Ayoub. *Les motifs des variétés rigides analytiques*. preprint, 2008.
- [2] J. Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*. to appear, 2008.
- [3] V. G. Berkovich. A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures. In *Algebra, Arithmetic and Geometry - Manin Festschrift (to appear)*. Boston: Birkhäuser.
- [4] V. G. Berkovich. *Spectral theory and analytic geometry over non-archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. AMS, 1990.
- [5] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Publ. Math., Inst. Hautes Étud. Sci.*, 78:5–171, 1993.
- [6] V. G. Berkovich. Vanishing cycles for formal schemes II. *Invent. Math.*, 125(2):367–390, 1996.
- [7] V. G. Berkovich. Smooth  $p$ -adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.
- [8] V. G. Berkovich. An analog of Tate's conjecture over local and finitely generated fields. *Int. Math. Res. Not.*, 2000(13):665–680, 2000.
- [9] P. Berthelot. Cohomologie rigide et cohomologie rigide à supports propres. *Prepublication, Inst. Math. de Rennes*, 1996.
- [10] E. Brieskorn. Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscr. Math.*, 2:103–161, 1970.
- [11] B. Conrad. Irreducible components of rigid spaces. *Ann. Inst. Fourier*, 49(2):473–541, 1999.
- [12] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Publ. Math., Inst. Hautes Étud. Sci.*, 82:5–96, 1995.
- [13] J. Denef and F. Loeser. Motivic Igusa zeta functions. *J. Algebraic Geom.*, 7:505–537, 1998, arxiv:math.AG/9803040.
- [14] J. Denef and F. Loeser. Geometry on arc spaces of algebraic varieties. *Progr. Math.*, 201:327–348, 2001, arxiv:math.AG/0006050.
- [15] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*, volume 35 of *Ergebnisse der Math. und ihrer Grenzgebiete*. Springer-Verlag, 1967.

- [16] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings 1*, volume 339 of *Lecture Notes in Mathematics*. Springer-Verlag, 1973.
- [17] V. Navarro Aznar. Sur la théorie de Hodge-Deligne. *Invent. Math.*, 90:11–76, 1987.
- [18] V. Navarro Aznar. Sur les structures de Hodge mixtes associées aux cycles évanescents. In *Hodge theory, Proc. U.S.-Spain Workshop, Sant Cugat/Spain 1985*, volume 1246 of *Lect. Notes Math.*, pages 143–153, 1987.
- [19] J. Nicaise. A trace formula for rigid varieties, and motivic Weil generating series for formal schemes. *to appear in Math. Ann.*
- [20] J. Nicaise and J. Sebag. Invariant de Serre et fibre de Milnor analytique. *C.R.Ac.Sci.*, 341(1):21–24, 2005.
- [21] J. Nicaise and J. Sebag. The motivic Serre invariant, ramification, and the analytic Milnor fiber. *Invent. Math.*, 168(1):133–173, 2007.
- [22] J. Nicaise and J. Sebag. Rigid geometry and the monodromy conjecture. In D. Chéniot et al., editor, *Singularity Theory, Proceedings of the 2005 Marseille Singularity School and Conference*, pages 819–836. World Scientific, 2007.
- [23] C. Peters and J.H.M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Berlin: Springer, 2008.
- [24] J.H.M. Steenbrink. Limits of Hodge structures. *Invent. Math.*, 31:229–257, 1976.
- [25] J.H.M. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In P. Holm, editor, *Real and complex Singularities, Proc. Nordic Summer Sch., Symp. Math., Oslo 1976*, pages 525–563. Alphen a.d. Rijn: Sijthoff & Noordhoff, 1977.
- [26] M. Temkin. Desingularization of quasi-excellent schemes in characteristic zero. *to appear in Adv. Math.*, arXiv:math/0703678.
- [27] A. Thuillier. Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels. *Manuscr. Math.*, 123(4):381–451, 2007.

UNIVERSITÉ LILLE 1, LABORATOIRE PAINLEVÉ, CNRS - UMR 8524, CITÉ SCIENTIFIQUE, 59655  
 VILLENEUVE D’ASCQ CÉDEX, FRANCE  
*E-mail address:* johannes.nicaise@math.univ-lille1.fr